Nonlinear Vibration of a One-Degree of Freedom Shape Memory Alloy Oscillator: A numerical-experimental Investigation

Dimitris C. Lagoudas* and Luciano G. Machado†
Texas A&M University, College Station, TX 77843-34141, USA
Magdalini Z. Lagoudas‡
Spacecraft Technology Center, College Station, TX 77843-3118, USA

Pseudoelastic Shape memory alloys (SMAs) are very attractive for passive vibration control due to their ability to sustain and retrieve large amounts of strain, dissipate high levels of energy and provide a restoring force to the system. They can be effectively used to attenuate vibrations of a primary system by introducing variable stiffness, and providing additional energy dissipation due to hysteresis. Motivated by these properties, this paper presents a dynamical investigation of a passive damping device, where the main elements are pseudoelastic SMA wires. The device, a mass connected to a frame by two SMA wires, was subjected to a series of continuous sinusoidal acceleration functions in the form of a sine sweep. Frequency responses and transmissibility of the device were analyzed for the case where the SMA wires were pre-strained at 4.0% of its original length. The temperatures of the wires throughout the dynamic tests were also recorded. In addition, numerical simulations of a single-degree of freedom SMA oscillator were conducted to corroborate experimental results. A thermodynamical constitutive model for SMAs was used to simulate constitutive pseudoelastic response of the SMA elements.

I. Introduction

For the past ten years, Shape Memory Alloys (SMAs) have attracted much attention as potential materials for the use in either seismic protection devices, passive or active vibration isolation systems. Pseudoelastic SMAs are very attractive for passive vibration control, due to their ability to sustain and retrieve large amounts of strains, dissipate high levels of energy and provide a restoring force to the system.

SMA passive isolation devices can be effectively used to attenuate vibrations of a primary system by introducing variable stiffness, as a consequence of stress-induced phase transformation, and providing additional energy dissipation due to their pseudoelastic hysteresis. Variable stiffness may shift the natural frequency away from the resonance, while energy dissipation largely reduces the transmissibility at resonance, where large deformations and displacements occur. SMA damping is variable in the sense that more energy will be dissipated through its hysteresis as the response amplitude increases.

Several references have investigated the vibration isolation capabilities of the SMAs. Salich et al. addressed the vibration suppression of one-story building structures reinforced with SMA diagonal brace wires. Two SMA constitutive models, one proposed by Lexcellent and Bourbon, and the other by Brinson were used for the numerical simulation. These two models were experimentally validated and the response of a structure equipped with SMA tendon wires was predicted and compared with the responses of structures...
using steel tendon wires and no tendons at all. Bartera and Giacchetti\textsuperscript{5} performed an experimental investigation on the dynamic response of a reinforced concrete structural frame with added stiffness and damping provided by dissipative bracing systems. The authors compared the dynamical response of the bare frame with two different bracing dissipative systems, one composed of pre-tensioned SMA wires, and the other of a high damping rubber pad.

Khan \textit{et al.}\textsuperscript{6,7} investigated the pseudoelastic response of SMAs on passive vibration isolation through numerical simulation and experimental correlation. A SMA simplified model and an empirical model based on system identification (Preisach model) were adapted to simulate pseudoelastic behavior of SMAs. An extensive parametric study on a nonlinear hysteretic dynamic system, representing an actual SMA damping and passive prototype device was conducted.

Lacarbonara \textit{et al.}\textsuperscript{8} numerically investigated the nonlinear responses and bifurcations of a one-degree of freedom shape memory oscillator. A thermomechanical model based on the work by Ivshin and Pence\textsuperscript{9} was utilized to describe the nonlinear constitutive behavior of the shape memory element of the oscillator. It was shown that a rich class of solutions, including discontinuity of frequency responses, quasi-periodicity and chaos could arise in nearly adiabatic conditions. Machado and Lagoudas\textsuperscript{10} evaluated the nonlinear dynamics of a one-degree of freedom shape memory alloy oscillator, with a special attention to chaotic responses. The simplified SMA model proposed by Khan \textit{et al.}\textsuperscript{6} was used to simulate the nonlinear behavior of the shape memory alloy. Phase Space Plots, Poincar Maps, Bifurcation Diagrams and Fast Fourier Transforms (FFT) were used to classify different types of motion of the oscillator. The analysis has shown that both periodic and chaotic motion can be observed.

This paper investigates the nonlinear dynamics of a passive damping vibration device, where the main elements are SMA wires. The device was subjected to a series of continuous sinusoidal acceleration functions in the form of a sine sweep. Frequency responses and transmissibility of the device are analyzed for the case where the SMA wires were pre-strained at 4.0% of its original length. In addition, the temperature of the wires was recorded during the dynamical tests, where a large variation was observed. Finally, numerical simulations of a single-degree of freedom SMA oscillator were conducted to corroborate the experimental results. The oscillator configuration was based on the SMA damping passive device. A thermodynamical constitutive model, proposed by Qidwai and Lagoudas\textsuperscript{11} is used to simulate the constitutive pseudoelastic response of the SMA elements. The numerical simulation was able to predict the discontinuity in the system response observed in the experimental sine sweep tests.

II. Experimental Investigation

Motivated by the unique properties of SMAs, an experimental setup was designed to investigate the passive vibration isolation capabilities of these materials. The SMA passive damping device (Fig. 1) is composed of a frame, two low-friction ball bearings, a mass of 0.6kg, and two pseudoelastic SMA wires of equal length (0.0762m) connecting the mass to the frame (both top and bottom). The two low-friction ball bearings prevent any lateral displacement or rotation of the mass. In addition, since wires do not support compressive loads, each SMA wire was preloaded with sufficient preload to assure that the wires remain in tension throughout the test. A screw assembled on the top plate of the frame provides the pre-tension of the wires.

Since the main objective of this paper is to investigate the SMA dynamic response, the frame was designed to be of high stiffness in order to avoid any resonance or structural mode of vibration that could contaminate the analysis of the SMA response within the frequency range of the experiment. For this reason, a dynamic analysis of the frame was performed in emphaBAQUS 6.4, where the eigenfrequencies and modes of vibrations of the frame were computed. Figure 2 presents the shape of the first four modes of vibration of the frame.

A. Material Selection and Characterization

The material selected for the experiment was a pseudoelastic NiTi wire, with 0.51mm of diameter. The basic requirement for the selection of the material was to exhibit pseudoelastic behavior at room temperature. Several thermomechanical tests were conducted with the purpose of characterizing and preparing the SMA wires for the vibration tests. At first, a sequence of thirty loading/unloading cycles was performed at constant temperature of 50°C (323.15K), at the rate of 0.035m/sec. The maximum stress value achieved
(a) SMA prototype device

(b) Schematic of the SMA prototype device

Figure 1. SMA prototype device

Figure 2. Modes of vibration of the device.
During loading was selected to be equal to 720MPa. The objective of this sequence of loading/unloading cycles was to stabilize the loop of hysteresis. Also, it is important to mention that even though the vibration tests were planned to be performed at room temperature, a temperature variation of the wires was expected. For this reason, the temperature of 50°C was selected for the realization of the loading/unloading cycles to assure that the wires retain their stable hysteresis properties at higher temperatures. Figure 3a presents the stress-strain curve of these cycles. After the thirty loading/unloading cycles, each SMA wire was cooled to room temperature, and then three single loading-unloading paths were performed at 25°C, 30°C, and 40°C, respectively (Fig. 3b).

![Stress-Strain curves](image)

**Figure 3. Stress-Strain curves.**

**B. Sine Sweep Vibration Tests**

The experiment consisted of exciting the SMA device (Fig. 1a) over a given frequency range by a series of continuous sinusoidal acceleration functions, in the form of a sine sweep. The amplitude of all sinusoidal acceleration functions was chosen to be a multiple of the acceleration of gravity, g. At first, four tests were performed for the frequency range of 32Hz to 256Hz (up sine sweep) with acceleration amplitudes of 0.5g, 1.0g, 2.0g, and 4.0g. Then, two tests followed for frequency range of 256Hz to 32Hz (down sine sweep) with acceleration amplitudes of 1.0g and 2.0g. For all sine sweep tests, the sweep rate was selected to be equal to 1.244Hz/sec, which resulted in total test time of 3.0min. The initial temperature of all tests was 25°C (295.15K), and both SMA wires were pre-strained at the level of 4.0% strain, so that the wires worked only in tension during the experiment. All vibration tests were contacted using a C126 shaker with a PUMA vibration control system by Spectral Dynamics.

Four accelerometers were used to record accelerations at different locations of the frame and on the shaker plate. The first accelerometer (control channel-1) was placed on the shaker plate, while the three others were positioned on different places of the frame, namely on the base plate (control channel-2), mass (control channel-3) and top plate (control channel-4) (see Fig. 1b). The accelerometers of the base and top plate of the frame measured the vibration of the frame and its possible influence on the mass dynamic response, while the accelerometer on the mass captures the effects of the SMA wires. The temperatures of the wires were also measured throughout the dynamical tests. One thermocouple was attached to each SMA wire (see Fig. 1 on the page before). A Labview program was used to record the temperatures during the dynamical tests.

As a standard procedure in vibration tests, the first sine sweep test is always a low amplitude acceleration input. The reason for this is to verify the safety and structural integrity of the device to be tested, before exposing it to higher accelerations. Therefore, a sine sweep test was performed at the level of 0.5g of acceleration, with a frequency range from 32Hz to 256Hz. Figure 4 presents the results of this sweep sine test. Figure 4a shows the acceleration responses measured by the four accelerometers. It can be noticed that the accelerations measured at the base (control channel-2) and top plate of the frame (control channel-4) coincide with the input acceleration of the shaker (control channel-1). This indicates that the frame natural frequency is outside the range of testing, as expected, and that there should not be a dynamic coupling between the frame and the mass. The mass acceleration reached its maximum value of 3.9g at the frequency
of 55Hz. The diagonal line represents the time scale of the experiment, with resonance occurring at 18.5 sec.

Figure 4. Sine sweep test. Amplitude of acceleration input = 0.5g, $T = 25^\circ C$

Figure 4b shows the transmissibility plot for the 0.5g sine sweep test. The transmissibility was computed as the ratio of the mass acceleration measured by accelerometer #3 (control channel-3) and the acceleration provided by the shaker measured by accelerometer #1 (control channel-1). From Fig. 4b it can be seen that the maximum transmissibility is 7.8 while the vibration isolation occurs for frequencies above 106 Hz.

Figure 5. Sine sweep tests (up) for 1.0g, 2.0g, and 4.0g of acceleration amplitude, $T = 25^\circ C$.

Figure 5 presents the results of the sine sweep tests for different levels of acceleration inputs, i.e. 1.0g, 2.0g, and 4.0g. These tests were performed for a frequency range of 32Hz to 256Hz. Figure 5a shows the variation of the mass acceleration with respect to the excitation frequency, while Figure 5b presents the transmissibility response.

By the analysis of Fig. 5a and Fig. 5b, it can be seen that, as the amplitude of the acceleration input increases, the resonance frequency of the system and the transmissibility decrease. Moreover, there is a
discontinuity (jump) in the system dynamic response, for the cases of 2.0g and 4.0g. The reduction of the resonance frequency of the system occurs as a consequence of the martensitic phase transformation that the SMA wires undergo during testing. The SMA wire stress-induced martensitic phase transformation results in lower tangent stiffness, which reduces the resonance frequency of the test article. In addition, the transmissibility reduction is related to the additional damping provided by the hysteresis loop. Higher amplitudes of the acceleration input result in higher additional damping. The transmissibility peak of 7.7 occurs at 54.0Hz for the test case of 1.0g, and then it decreases to 6.5 (at 54.8Hz) for 2.0g, and to 4.0 (at 41.0Hz) for 4.0g.

![Acceleration response](image1)

(a) Acceleration response

![Transmissibility response](image2)

(b) Transmissibility response

Figure 6. Sine sweep tests (down) for 1.0g and 2.0g of acceleration amplitude, $T=25^\circ C$.

The discontinuity of the system response is more evident for amplitudes of 2.0g and 4.0g, and it happens before the resonance frequency as shown in Figure 5. This occurs as a result of the nonlinear softening behavior of the SMA wires during the dynamical test. For the 2.0g case, the transmissibility response jumps from 4.2 to 5.0, while for 4.0g it jumps from 1.9 to 3.2. It can be also noticed that the frequency at which the transmissibility is less than one progressively decreases from 106.0Hz at 0.5g (Fig. 4) to 98.4Hz for 1.0g, 88.0Hz for 2.0g, and stays at 89.0Hz for 4.0g.

Next, two vibration tests were conducted with decreasing excitation frequency. The frequency was varied from 256Hz to 32Hz (down sine-sweep). Figure 6 shows the results of these tests for 1.0g, and 2.0g levels of acceleration amplitude of the sine sweep. Here, it is observed that martensitic phase transformation and hysteresis have similar effects as before. However, the discontinuities in the dynamic response of the system are more evident. For the case of 1.0g, the transmissibility value changes from 8.9 to 5.0, while for the case of 2.0g the transmissibility peak is largely reduced from 8.0 to 1.6.

Figure 7 presents SMA wire temperature data as a function of time, for all the sine sweep tests presented on Fig. 5 and Fig. 6. The initial temperature for all dynamic tests was 25°C. It can be noticed that the temperatures largely increase as a result of the SMA wire martensitic phase transformation during the experiments. The higher amplitude of acceleration input, the higher is the temperature variation.

![Temperature variation](image3)

Figure 7. Sine sweep tests - Temperature variation.
of the wires, denoting a very strong thermomechanical coupling.

In addition, the highest value of temperature for all sine sweep tests occurred after the test article was excited at its resonance frequency. Table 1 presents for each test the resonance frequency and the frequency at which the highest temperature was recorded.

Table 1. Comparison between Resonance and Maximum Temperature Frequencies

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>32 to 256Hz</td>
<td>1.0g</td>
<td>46.2 °C</td>
<td>21.2s</td>
<td>58.4Hz</td>
<td>17.8s</td>
<td>54.0Hz</td>
</tr>
<tr>
<td>32 to 256Hz</td>
<td>2.0g</td>
<td>49.8 °C</td>
<td>22.4s</td>
<td>59.8Hz</td>
<td>16.5s</td>
<td>52.8Hz</td>
</tr>
<tr>
<td>32 to 256Hz</td>
<td>4.0g</td>
<td>53.0 °C</td>
<td>16.0s</td>
<td>43.2Hz</td>
<td>7.2s</td>
<td>41.0Hz</td>
</tr>
<tr>
<td>256 to 32Hz</td>
<td>1.0g</td>
<td>35.8 °C</td>
<td>163.8s</td>
<td>52.0Hz</td>
<td>162.3s</td>
<td>54.5Hz</td>
</tr>
<tr>
<td>256 to 32Hz</td>
<td>2.0g</td>
<td>52.6 °C</td>
<td>171.0s</td>
<td>43.2Hz</td>
<td>172.3s</td>
<td>42.1Hz</td>
</tr>
</tbody>
</table>

At this point, it is important to compare some features related to the transmissibility of a linear vibration isolation system with the transmissibility of a vibration system composed with SMA wires. First of all, the transmissibility of a linear system is a single-valued curve, where there is no discontinuity (jump) present, and higher damping results in lower transmissibility values. However, the damping has no effect on the point where the effective isolation happens, nor on the value of the eigenfrequency. Also, there is no change in the temperature of the isolation system. The analysis of the experimental results of the passive SMA damping device has shown that the transmissibility curves present a discontinuity related to the hysteretic behavior of the SMA wires. The damping effect on the SMA system is variable, and it is a function of the area of the loop of hysteresis. Also, the stress-induced martensitic transformation that the SMA wires undergo reduces the resonance frequency of the system, and largely increases the temperature of the SMA wires.

### III. Numerical Simulations

#### A. One-Degree of Freedom Shape Memory Alloy Oscillator

The test article is modeled as a one-degree of freedom oscillator (Fig. 8) composed of a mass attached to two pseudoelastic SMA wire elements that are initially in austenitic phase. The configuration of this oscillator is based on the passive damping device presented before. The differential governing equation of motion of the mass of the oscillator is given by Eq. (1) below.

\[
m\ddot{x} = F^SMA_U - F^SMA_L
\]  

(1)

An important observation is that since the SMA is rate independent, the dissipation in the system is due to hysteresis, which is path dependent. As a consequence, no velocity dependent term due to rate dependent dissipation, such as in viscoelastic materials, is considered in Eq. (1), where \( m \) is the mass, \( \ddot{x} \) is the acceleration of the mass, \( F^SMA_U \) is the force exerted by the wire above the mass, and \( F^SMA_L \) is the force exerted by the wire below the mass.

Notice that if the SMA wires were replaced by elastic elements, then the two wires could be modeled as one equivalent wire and the wire force acting on the mass could be modeled as a single force (see also a discussion in Khan et al\(^7\)). However, in this case, the forces exerted by the SMA wires are dependent on the history of the displacement, , and thus the system cannot be modeled considering just one equivalent SMA.
wire. The forces exerted by the SMA wire above the mass and beneath the mass have, in general, different magnitudes, since the upper and the lower displacements have opposite histories and there is a pre-strain that is imposed on the wires at static equilibrium. Equation (2) below describe the wire length change as a function of the mass and base displacements, respectively, i.e.,

\[ \delta_L = -\delta_U = x(t) - y(t) \] (2)

where \( \delta_U \) is the upper wire displacement, and \( \delta_L \) is the lower wire displacement, The subscripts \( U \) and \( L \) denote upper and lower wires, respectively, while \( x(t) \) is the mass displacement, and \( y(t) \) is the base displacement. The system is harmonically excited by a base displacement in a sinusoidal form, as given by Eq (3).

\[ y(t) = L \sin(\omega t) \] (3)

\[ L = \frac{ag}{\omega^2} \] (4)

where \( \omega \) is the excitation circular frequency, and \( L \) (given by Eq. (4)) is the amplitude of the imposed displacement, given in terms of \( a \), the fraction of the gravity acceleration, \( g \).

B. Thermomechanical Constitutive Model for Shape Memory Alloys

Now that the dynamical problem is defined, a constitutive model for the SMA needs to be considered in order to simulate the constitutive behavior of the SMA wires.

The constitutive behavior of the SMA wires is simulated with the implementation of the thermomechanical constitutive model for SMAs proposed by Lagoudas and Qidwai.\textsuperscript{11} This model is based on the thermodynamic framework proposed by Boyd and Lagoudas,\textsuperscript{13} and it is able to capture the main characteristics of pseudoelastic SMA response. The constitutive model is presented briefly here. For more details about the constitutive model, the reader is referred to the work by Lagoudas and Qidwai.\textsuperscript{11}

Since the SMA wires can be modeled as one-dimensional elements, this work just utilizes a 1-D reduction of the constitutive model. Following the thermodynamic framework proposed by Boyd and Lagoudas,\textsuperscript{13} the constitutive model is formulated in terms of the Gibbs free energy,\textsuperscript{11} \( G \), where the independent state variables are stress, temperature, martensitic volume fraction, and transformation strain. The specific form of the total Gibbs free energy in one-dimensional case of a polycrystalline SMA, assuming linear thermoelastic response and non-linear transformation-hardening behavior, is given below;

\[ G(\sigma, T, \xi, \varepsilon^t) = -\frac{1}{2} S\sigma^2 - \frac{1}{\rho} \sigma (T - T_0) + c \left[ (T - T_0) - T\ln \left( \frac{T}{T_0} \right) \right] - s_0 T + u_0 + f(\xi) \] (5)

where \( \sigma \), \( \varepsilon^t \), \( \xi \), \( T \), and \( T_0 \) are the stress, transformation strain, martensitic volume fraction, temperature and reference temperature, respectively. The material constants \( S, \alpha, \rho, c, s_0, u_0 \) are the effective compliance, effective thermal expansion coefficient, density, effective specific heat, effective specific entropy at the reference state, and the effective specific internal energy at the reference state, respectively. The form of the hardening function, \( f(\xi) \), will be given latter. The effective material properties can be defined in terms of the martensitic volume fraction, \( \xi \), by the rule of mixtures, as follows:

\[ S = S^A + \xi (S^M - S^A) ; \quad c = c^A + \xi (c^M - c^A) ; \quad s_0 = s_0^A + \xi (s_0^M - s_0^A) ; \]

\[ u_0 = u_0^A + \xi (u_0^M - u_0^A) ; \quad \alpha = \alpha^A + \xi (\alpha^M - \alpha^A) \]

The Gibbs free energy in Eq. (5) is related to the internal energy, \( u \), through the Legendre Transformation:

\[ u(\sigma, s, \xi, \varepsilon^t) = G + Ts + \frac{1}{\rho} \sigma \varepsilon \] (7)

where \( \varepsilon \) is the total strain.
The total strain, which can be obtained by following a standard thermodynamic procedure where and are substituted into the first and second law of thermodynamics, is defined below:  
\[ \varepsilon = -\rho \frac{\partial G}{\partial \sigma} = S\sigma + \alpha (T - T_0) + \varepsilon^t \]  

(8)

One assumption that can be made for the case of phase transformation without reorientation is that any change in the current state of the system is only possible due to a change in the martensitic volume fraction, and that any other internal state variable evolves with it. Therefore, the relation between the evolution of the transformation strain and the evolution of the martensitic volume fraction during forward and reverse transformation is expressed by:

\[ \dot{\varepsilon}^t = \Lambda \dot{\xi} \]  

(9)

where the transformation tensor, \( \Lambda \), is defined by the following expression:

\[ \Lambda = H \text{sgn} (\sigma) \]  

(10)

The Thermodynamic Force conjugated to \( \xi \) is defined by the relation below.

\[ \pi = H |\sigma| + \frac{1}{2} \Delta S\sigma^2 + \Delta \alpha T (T - T_0) + \rho c \left[ (T - T_0) - T \ln \left( \frac{T}{T_0} \right) \right] + \rho \Delta s_0 T + \rho u_0 - \frac{\partial f(\xi)}{\partial \xi} \]  

(11)

The terms defined with the prefix \( \Delta \) in Eq. (11) denotes the difference of the quantity between the martensitic and austenitic phases.

A transformation function, \( \Phi \), can be defined in terms of the thermodynamic force, \( \pi \), as follows:

\[ \Phi = \begin{cases} \pi - Y^*; & \dot{\xi} > 0 \\ -\pi - Y^*; & \dot{\xi} < 0 \end{cases} \]  

(12)

where \( Y^* \) is the measure of internal dissipation due to phase transformation. Constrains on the evolution of the martensitic volume fraction are expressed by the Kuhn-Tucker conditions as:

\[ \xi = 0; \quad \Phi (\sigma, T, \xi) \leq 0; \quad \Phi \dot{\xi} = 0 \]  

\[ \xi = 0; \quad \Phi (\sigma, T, \xi) \leq 0; \quad \Phi \dot{\xi} = 0 \]  

(13)

The hardening function \( f(\xi) \) is responsible for the transformation-induced strain hardening in the SMA material. Different forms of the transformation hardening function can be selected according to different constitutive models.\(^\text{11}\) To obtain the constitutive model presented by Boyd and Lagoudas\(^\text{13}\) the function \( f(\xi) \) has the following form:

\[ f(\xi) = \begin{cases} \frac{1}{2} \rho b^M \xi^2 + (\mu_1 + \mu_2) \xi; & \dot{\xi} > 0 \\ \frac{1}{2} \rho b^A \xi^2 + (\mu_1 - \mu_2) \xi; & \dot{\xi} > 0 \end{cases} \]  

(14)

where \( \rho b^M, \rho b^A, \mu_1, \mu_2 \) are transformation strain hardening materials constants.

1. Determination of the material constants used in the constitutive model

Now, it is necessary to determine the material constants of the SMA wires in order to calibrate the model. Since the initial temperatures of all vibration tests were measured to be 25 °C, this temperature will be used as a reference temperature \( (T_0) \). Figure 3c shows the stress strain curve for the temperature of 25 °C, where the Young’s modulus of both austenitic \( (E^A) \), and martensitic phase \( (E^M) \), and the value of maximum transformation strain \( (\varepsilon^t) \) were obtained.

It is important to mention that this model assumes that the transformation temperature curves of the stress-temperature space are straight lines and have the same slope, even though this is not always true.
for different stiffness between austenite and martensite. The slope of the stress-temperature curves can be computed by defining the stress values for which the martensitic phase transformations (forward and reverse) start and end, i.e., from the different temperature tests (20°C, 30°C, and 40°C) presented on Fig. 3b. Then, with these three sets of stress values, four different straight lines can be interpolated in the stress-temperature space, and thus the transformation temperatures and their slope can be defined. The transformation temperatures at zero-stress, i.e., M₀f, M₀s, A₀f, and A₀s, can be approximately obtained by computing the intersection points of the stress-temperature curves with the temperature axis. In fact, the transformation temperatures calculated are not the same as those obtained by a differential calorimetry test. However, since the temperature range of interest on this paper is for temperatures higher than austenitic finish temperature, this is a valid assumption. Table 2 presents the values of the material constants.

<table>
<thead>
<tr>
<th>Table 2. Material Constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>E^A = 33.0 · 10⁹ Pa</td>
</tr>
<tr>
<td>E^M = 15.0 · 10⁹ Pa</td>
</tr>
<tr>
<td>H = 0.023</td>
</tr>
<tr>
<td>T₀ = 25°C</td>
</tr>
<tr>
<td>M₀f = -46°C</td>
</tr>
<tr>
<td>A₀s = -12°C</td>
</tr>
</tbody>
</table>

The entropy difference ρΔs₀ per unit of volume between the phases can be determined by the slopes of the stress-temperature transformation curves. The slopes of the transformation curves can be analytically determined through the following equation:

\[
\frac{dσ}{dT} = -\frac{ρΔs₀ + Δασ}{H + Δσσ + Δα (T - T₀)}
\]  

Substituting zero stress and neglecting the thermal term in the denominator of the above equation, the slope \( \frac{dσ}{dT} \) of these curves is found to be \(-\frac{ρΔs₀}{H} \)

Table 3 presents the expressions describing these model parameters.

<table>
<thead>
<tr>
<th>Table 3. Model Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>ρb^A = -ρΔs₀ (A₀f - A₀s)</td>
</tr>
<tr>
<td>ρb^M = -ρΔs₀ (M₀s - M₀f)</td>
</tr>
<tr>
<td>γ' = ( \frac{1}{2} ρΔs₀ (M₀s + A₀f) = ρΔu₀ + μ₁ )</td>
</tr>
<tr>
<td>Y* = (-\frac{1}{2} ρΔs₀ (A₀f - M₀s) - \frac{1}{4} ρΔs₀ (M₀s - M₀f - A₀f + A₀s) )</td>
</tr>
<tr>
<td>μ₂ = ( \frac{1}{4} (ρb^A - ρb^M) )</td>
</tr>
</tbody>
</table>

Figure 9 presents the comparison of the numerical implementation of the SMA model with a stress-strain curve obtained from a tensile test of the same SMA wire utilized in the sine sweep tests, for a temperature of 25°C. It is important to mention that due to the polycrystalline nature of the SMA wire, the start and end points of the forward and reverse martensitic transformation of the experimental stress-strain curve are not well defined, as opposed to the model prediction. In fact, this is one of the limitations of this constitutive SMA model, which predicts sharp corners at the beginning and end of the martensitic transformations. Moreover, it can be noticed that there is still some residual strain after the fully unloading of the material.
and that the elastic loading and unloading path of the martensitic phase do not coincide. The residual strain is due to plastic strains that are developed during the loading/unloading cycle, which is another limitation of the model, since it cannot capture plastic strains. The discrepancy between the loading and unloading of the martensitic phase happens due to the polycrystalline nature of the SMA, where grains have different orientation, and start and finish the transformation at different times. If the loading were allowed to reach higher values of force, the material would completely transform and this effect would be minimized.

It is also important to mention that since the sine sweep tests were realized for a 4.0% pre-strain of the SMA wires, the transformation strain hardening material constants were chosen such that the level of 4.0% strain lies at the end of the forward martensitic transformation. It is expected that for 4.0% pre-strain the model will predict level of stress of the martensitic transformation lower than actual values.

C. Numerical Implementation of the Constitutive Model

The numerical implementation of the constitutive model in this work follows the same procedure described in Qidwai and Lagoudas. Basically, given an increment of strain and temperature, the incremental form of the SMA constitutive model provides an increment of stress as an outcome. The increment of stress is calculated by implementing the Return Mapping Algorithm.

The Return Mapping Algorithm solves the thermoelastic-transformation problem defined by Eq. (8), Eq. (9), and Eq. (11), by dividing it into two problems using an additive split. At first, a thermoelastic prediction problem assuming that the increment of the transformation strain is zero is tried. If the predicted thermoelastic state violates the consistency condition, in other words, if it lies outside the transformation surface ($\Phi > 0$), a transformation correction problem takes place to restore the consistency condition. Qidwai and Lagoudas presents two different return mapping algorithms for the correction step, namely the Closest Point Projection Return Mapping Algorithm and the Cutting Plane Return Mapping Algorithm. This work uses the Cutting Plane Return Mapping Algorithm as the corrector algorithm. The main idea of the Cutting Plane algorithm is that it relies on integrating the transformation correction equations in an explicit manner and on linearizing the consistency condition. The Newton’s iteration method is applied to calculate the increment of martensitic volume fraction. It is important to mention that both return mapping algorithms utilize the same thermoelastic prediction step.

D. Dynamical Response of the SMA Oscillator

Now that the equations of motion of the SMA oscillator and the thermomechanical constitutive model for the SMA wires have already been defined, we proceed by integrating numerically Eq. (1), and thereby predicting the dynamical response of the oscillator. Since the response of the SMA oscillator is highly nonlinear, an efficient and reliable numerical method should be employed to assure stability and convergence of the solution. For this reason, Newmark integration scheme is used to compute the time response of the system.

Originally, Newmark proposed as an unconditional stable scheme, the case where the weight parameters $\alpha$ and $\gamma$ are equal to 0.25 and 0.5, respectively. However, in this work, time integration is performed by Galerkin Method, a variant of the Newmark scheme, where $\alpha$ and $\gamma$ are defined to be equal to 0.5 and 1.6, respectively. Time step and weighting parameters are chosen in order to ensure the stability and convergence of the solution. According to the Newmark scheme, the function and its derivatives are approximated as
follows:

\[
x_{n+1} = x_n + \Delta t \dot{x}_n + \frac{1}{2} \Delta t^2 \ddot{x}_n + \frac{1}{2} \Delta t^2 \dddot{x}_n
\]

After re-arranging some terms of Eq. (16), one can easily show that:

\[
\dddot{x}_n = a_3 x_{n+1} - G_n
\]

where:

\[
G_n = a_3 x_n - a_4 \dot{x}_n - a_5 \ddot{x}_n
\]

Substituting Eq. (17) into Eq. (1), one can easily find the relation

\[
x_{n+1} = \frac{\tilde{F} + mG_n}{a_3 m}
\]

where

\[
\tilde{F} = F^{SMA}_U - F^{SMA}_L
\]

The expression for acceleration can be obtained from Eq. (17), while the expression for velocity is given below:

\[
\dot{x}_{n+1} = x_n + a_2 \dot{x}_n + a_1 \ddot{x}_n + 1
\]

where

\[
a_1 = \alpha \Delta t; \quad a_2 = (1 - \alpha) \Delta t
\]

Notice that \(\tilde{F}\) is function of the forces exerted by the upper and lower wires. These forces are functions of the variable tangent stiffness of the SMA wires that are displacement history dependent. Therefore, the actual solution of Eq. (20) is computed through an iterative scheme. For each time interval the displacement of both SMA wires is calculated. Then, the displacement history is used as input for the cutting plane return mapping algorithm, which resolves the nonlinear behavior of the material and updates the value of the tangent stiffness and the value of the forces exerted by the SMA wires. The displacement of the previous converged time step, \(x_{n+1}^k\), is used as a initial condition for the actual time step \(x_{n+1} = x_n\). Eq. (24) is computed until convergence is reached \(|x_{n+1}^{(k+1)} - x_n| < tol = 1.0e - 6\).

Next, the numerical simulation results will be compared to data obtained from the sine sweep tests. In order to consider the temperature change effect observed in the experimental sine sweep tests, a polynomial curve of 9th order was fit to each set of temperature-frequency test data. Figure 10 presents the temperature curve fit for two sine sweep tests, 1:0g and 2:0g, for the frequency interval of 32Hz to 256Hz.

Figure 11 presents the comparison among the transmissibility curves obtained from numerical simulation and data obtained from the sine sweep tests for two frequency intervals, 32Hz to 256Hz (up sine sweep) and 256Hz to 32Hz (down sine sweep), for acceleration input amplitudes of 1:0g (Fig. 11a) and 2:0g (Fig. 11b).

In Fig. 11a, the resonance frequency interval predicted by the numerical simulations is in good agreement with both sets of test data (up and down sine sweeps). Moreover, the numerical simulation was able to predict the discontinuity (jump) of the transmissibility curve that was previously observed in the experimental results. However, some discrepancies can also be noticed in this comparison. Even though the simulations
were able to capture the discontinuity in the frequency response, the peak value of the numerical transmissibility curve is lower than the actual value. This effect is an indication that the model may be introducing more damping than really exists. In addition, since the 4% pre-strain is at the end of the stress-strain curve (Fig. 9), the analytical model is expected to differentiate the most from the experimental data. The predicted frequency for transmissibility of one is lower than both experimental curves. The discontinuity of the numerical transmissibility curve happens between 50Hz and 51Hz of frequency and the transmissibility value jumps from 4.9 to 7.2.

Similarly Fig. 11b shows that the numerical simulation for 2.0g was able to capture the discontinuity in the frequency response. Here, the predicted frequency at which the transmissibility is equal to one is much closer to the actual experimental results. One can also notice that there still exists some discrepancy among the simulation and the numerical results, also related with modeling damping effects. The peak value of the numerical transmissibility curves lies in-between the two peaks from the sine sweep tests (up and down). The discontinuity of the numerical transmissibility curve happens between 43Hz and 44Hz of frequency and the transmissibility value jumps from 2.5 to 6.9. Moreover, the analytical model predicts a transmissibility curve with a plateau between the frequencies of 56Hz and 72Hz, while such a plateau was not observed in the experimental results.
Next, a set of analytical results was generated for a single frequency of excitation, 1.0g, and the results are shown in Fig. 12, Fig. 13, Fig. 14 and Fig. 15. For all these simulations, the temperatures of the SMA wires were considered to be equal to the temperatures estimated by the curve fitting. Moreover, these results were considered for the steady state solution. Figure 12 shows the results for case where the excitation frequency was of 50Hz that is the last value of frequency before the jump in the transmissibility curve. Even though the SMA wires were considered to be pre-strained at 4.0%, which lies at the end of the phase transformation, the oscillator behaves linearly after the transient dies out, where the elastic modulus of the wires happens to be equal to the martensite elastic modulus. Figure 12a, and Fig. 12b shows the stress-strain curves for the lower and the upper wire, respectively, while Fig. 12c presents the phase space diagram.

![Stress-strain curves and phase space diagram for 1.0g of acceleration amplitude and 50Hz of excitation frequency](image)

**Figure 12. Numerical simulation for 1.0g of acceleration amplitude, and 50Hz of excitation frequency.**

Figure 13 shows the results for the case where the excitation frequency was 51Hz, which is the first frequency test data point after the jump. It can be concluded now that the discontinuity of the transmissibility happens due to the appearance of the loop of hysteresis, even though the loop is not completed. The loop of hysteresis is responsible for introducing additional damping to the system, where the amount of damping is directly related to the area of the loop of hysteresis.

![Stress-strain curves and phase space diagram for 1.0g of acceleration amplitude and 51Hz of excitation frequency](image)

**Figure 13. Numerical simulation for 1.0g of acceleration amplitude, and 51Hz of excitation frequency.**

Figure 14 presents the results for an excitation frequency of 56Hz. Similarly to the case of Fig. 13, one can notice that there still exists the loop of hysteresis, for the steady state solution. However, the area of the loop is reduced, compared to the loop of Fig. 13. On the other hand, Fig. 15 presents results for an excitation frequency of 64Hz. It can be noticed that the SMA oscillator returns to the undamped linear behavior, without any loop of hysteresis.

![Stress-strain curves and phase space diagram for 1.0g of acceleration amplitude and 56Hz of excitation frequency](image)

**Figure 14. Numerical simulation for 1.0g of acceleration amplitude and 56Hz of excitation frequency.**

Figure 15 presents results for an excitation frequency of 56Hz. Similarly to the case of Fig. 13, one can notice that there still exists the loop of hysteresis, for the steady state solution. However, the area of the loop is reduced, compared to the loop of Fig. 13. On the other hand, Fig. 15 presents results for an excitation frequency of 64Hz. It can be noticed that the SMA oscillator returns to the undamped linear behavior, without any loop of hysteresis.

Figure 16, Fig. 17, Fig. 18, Fig. 19 and Fig. 20 present simulations for a single excitation frequency with acceleration input amplitude of 2.0g. Here again the temperatures of the SMA wires were considered being equal to the temperatures predicted by the curve fitting, and the results are presented only for the steady state solution. Figure 16 shows the results for a frequency of 43Hz. This frequency is right before the jump of the transmissibility curve. Again, the behavior of the oscillator is equivalent to an undamped linear oscillator. On the other hand, for the case of 44Hz (Fig. 17), the loop of hysteresis appears again.
This time, the linear part of the loop related to the austenitic phase reach much higher levels of stress than observed for 1.0g.

As we continue to raise the value of the excitation frequency, the area of the loop of hysteresis gets smaller. This can be verified by the results presented in Fig. 18, for an excitation frequency of 48Hz.

Figure 19 shows the results for 60Hz. This frequency falls on the plateau of the transmissibility curve. By the analysis of the results one can see that there still exists the loop of hysteresis, but the solution never reaches the end of the transformation. In other words, the oscillator wires are never in a pure austenitic state.

Finally, Fig. 20 shows the result for 75Hz. This frequency is located after the end of the plateau. As
expected, the oscillator behaves linearly again.

Figure 17. Numerical simulation for 2.0g of acceleration amplitude, and 44Hz of excitation frequency.

Figure 18. Numerical simulation for 2.0g of acceleration amplitude, and 48Hz of excitation frequency.

Figure 19. Numerical simulation for 2.0g of acceleration amplitude, and 60Hz of excitation frequency.

IV. Conclusion

This paper presented a numerical-experimental investigation of the forced vibration of a mass constrained by two pre-strained SMA wires. A series of sine sweep vibration tests were conducted at different acceleration levels. The frequency response and transmissibility of the device for SMA wires pre-strained at
4.0% strain were obtained. The experimental results have shown that the transmissibility curves present a discontinuity (jump) related to the nonlinear damping introduced by the hysteresis that the SMA wires undergo. In addition, a temperature variation of the SMA wires was measured, related to the martensitic phase transformation.

In order to corroborate the experimental results, numerical simulations of a one-degree of freedom oscillator were conducted. The oscillator configuration was based on the experimental device. The behavior of the SMA wires was simulated through the implementation of a thermomechanical constitutive model. The numerical implementation of the model was able to predict the discontinuity in the transmissibility response and approximate the peak value of the transmissibility curve. The measured temperature variation of the SMA wires was considered as input in the model and it was found to have some impact on the transmissibility curves.

V. Acknowledgements

The authors acknowledge Brazilian Agency CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior) and the Texas Institute for Intelligent Bio-Nano Materials and Structures for Aerospace Vehicles (TIiMS) for the financial support; also the support of the Spacecraft Technology Center (S.T.C.) for facilitating the realization of the vibration tests is acknowledge.

References


