Modeling of transformation-induced plasticity and its effect on the behavior of porous shape memory alloys.  
Part I: constitutive model for fully dense SMAs

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Abstract

A three-dimensional rate-independent thermomechanical constitutive model for fully dense shape memory alloys (SMAs) is developed in this Part I of a two-paper series. The model accounts for simultaneous development of transformation and plastic strains during stress-induced martensitic phase transformation, as well as for the evolution of shape and size of the hysteresis with repeated transformation cycling. A detailed procedure for the estimation of the material parameters is presented and a parametric study is also performed. The model is numerically implemented using return mapping algorithms. Using experimental data for SMA strips and large diameter wires, the material parameters for the model are estimated and the modeling results are with experiments. The model developed in Part I will be used in Part II [P.B. Entchev, D.C. Lagoudas, Mech. Mater., this issue] in a micromechanical averaging scheme to model the behavior of porous SMAs undergoing mechanical cyclic loading.

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1. Introduction

Since the discovery of the shape memory alloys (SMAs) in the 1960s researchers have been investigating both experimental aspects of their behavior as well as their constitutive modeling. While most of the issues associated with the thermomechanical response of SMAs have been addressed, there are still questions that remain open. One such open question is associated with the development of plastic strains during cyclic phase transformation. As indicated in the literature review below, this problem has been addressed for the case of one-dimensional SMA actuators (wires) undergoing cyclic temperature-induced phase transformation. However, a three-dimensional formulation is still lacking.

This problem resurfaces when dealing with porous SMAs undergoing cyclic phase transformation. It has been experimentally observed that large amount of plastic strain accumulates during such loading and it is very desirable to have a...
model, which is able to capture this effect. When modeling porous SMAs using micromechanical methods, one has to first establish an accurate model for the dense SMA matrix, which forms the walls and the connecting struts of the porous SMA. Due to the inherent three-dimensional stress state that develops during loading of porous SMAs, their successful modeling requires a three-dimensional model for the dense SMA matrix, that accounts for plastic strain development during cyclic loading.

Therefore, the focus of the current research effort will be to establish a three-dimensional model for fully dense SMAs with simultaneous evolution of transformation and plastic strains. Special attention will also be paid to the estimation of the material parameters required for the modeling of stress-induced phase transformation. The development of this model for dense SMAs will be presented in the first part of this two-part paper. It will then be used in the second part (Entchev and Lagoudas, to be published) to obtain the effective properties of porous SMAs using micromechanical averaging techniques.

1.1. Properties and applications of dense SMAs

SMAs are metallic alloys which can recover permanent strains when they are heated above a certain temperature. The key characteristic of all SMAs is the occurrence of a martensitic phase transformation. The martensitic transformation is a shear-dominant diffusionless solid-state phase transformation occurring by nucleation and growth of the martensitic phase from a parent austenitic phase. When an SMA undergoes a martensitic phase transformation, it transforms from its high-symmetry, usually cubic, austenitic phase to a low-symmetry martensitic phase, such as the monoclinic variants of the martensitic phase in a NiTi SMA.

The SMAs can be fabricated either as single crystals, such as some of the copper-based SMAs, or polycrystal, such as NiTi-based, iron-based and copper-based SMAs. While single crystal SMAs can be used for detailed analysis of the nature of the martensitic phase transformation, their use in applications is limited by the high manufacturing costs. Thus, since most practical applications involve polycrystalline SMAs, the current work will focus on the description of polycrystalline SMAs.

The key effects of SMAs associated with the martensitic transformation, which are observed according to the loading path and the thermo-mechanical history of the material are: pseudoelasticity, one-way shape memory effect and two-way shape memory effect.

An SMA exhibits the Shape Memory Effect (SME) when it is deformed while in the martensitic phase and then unloaded while still at a temperature below $M^\text{f}$. If it is subsequently heated above $A^\text{f}$ it will regain its original shape by transforming back into the parent austenitic phase. The pseudoelastic behavior of SMAs is associated with recovery of the transformation strain upon unloading and encompasses both superelastic and rubberlike behavior (Otsuka and Wayman, 1999).

The superelastic behavior is observed during loading and unloading above $A^\text{s}$ and is associated with stress-induced martensite and reversal to austenite upon unloading. When the loading and unloading of the SMA occurs at a temperature above $A^\text{hs}$, partial transformation strain recovery takes place. When the loading and unloading occurs above $A^\text{hf}$, full recovery upon unloading takes place.

The superelastic behavior described above constitutes an approximation to the actual behavior of SMAs under applied stress. In fact, only a partial recovery of the transformation strain induced by the applied stress is observed. A small residual strain remains after each unloading. Experimental results on the behavior of SMAs undergoing cyclic loading have been presented by McCormick and Liu (1994), Strnad et al. (1995a,b), Lim and McDowell (1994, 1999), Bo and Lagoudas (1999b), Kato et al. (1999) and Sehitoglu et al. (2001), among others. The thermo-mechanical cycling of the SMA material results in what is called training process. Different training sequences can be used, i.e., by inducing a non-homogeneous plastic strain at a martensitic or austenitic phase; by aging under applied stress, in the austenitic phase, in order to stabilize the parent phase, or in the martensitic phase, in order to create a precipitant phase (Ni–Ti alloys); by thermo-
mechanical, either superelastic or thermal cycles (Manach and Favier, 1993; Hebda and White, 1995; Liu and McCormick, 1990; Miller and Lagoudas, 2001). The main result of the training process is the development of Two-Way Shape Memory Effect (TWSME). In the case of TWSME, a shape change is obtained both during heating and cooling. Another effect of the training cycle is the development of macroscopically observable plastic strain. The magnitude of this strain is comparable to the magnitude of the recoverable transformation strain. The training also leads to secondary effects, like change in the transformation temperatures, change in the hysteresis size and decrease in the macroscopic transformation strain. These effects are similar to those observed during thermomechanical fatigue tests (Ronget al., 2001). It is important to define optimal conditions of training, because an insufficient number of training cycles produces a non-stabilized two-way memory effect and over-training generates unwanted effects that reduce the efficiency of training.

1.2. Review of thermomechanical constitutive models for SMAs

During the last two decades the area of constitutive modeling of SMAs has been a topic of many research publications and significant advancements have been reported. The majority of the constitutive models reported in the literature can be formally classified to belong to one of the two groups: micromechanics-based models and phenomenological models. Representative works from both of these groups are reviewed below.

The essence of the micromechanics-based models is in the modeling of a single grain and further averaging of the results over a representative volume element (RVE) to obtain a polycrystalline response of the SMA. Such models have been presented in the literature by different researchers. As an example, the micromechanics-based model based on the analysis of phase transformation in single crystals of copper-based SMAs has been presented by Patoor et al. (1996). The behavior of a polycrystalline SMA is modeled by utilizing the model for single crystals and using the self-consistent averaging method to account for the interactions between the grains. A micromechanical model for SMAs which is able to capture different effects of SMA behavior such as superelasticity, shape memory effect and rubber-like effect has been presented by Sun and Hwang (1993a,b). In their work, the evolution of the total transformation strain in each grain is considered. Then the evolution of the martensitic volume fraction is obtained by balancing the internal dissipation during the phase transformation with the external energy output. One of the recent micromechanical models for SMA has been presented by Gao et al. (2000a,b), where the evolution of each martensitic variant has been considered. The advantage of the micromechanics models is their ability to predict the effective material response using only crystal lattice parameters and information from the martensitic transformation at the crystalline and grain levels. Thus, their use provides valuable insight on the phase transformation process at the crystal level. Their disadvantage, however, is in the large number of numerical computations required to be performed, and the complexity of the micro-to-macro transition. Thus the use of such models for directly modeling the overall response of SMA structures is rather limited.

In the case of the phenomenological models, a macroscopic free energy function that depends on internal variables and their evolution equations are usually postulated and used in conjunction with the second law of thermodynamics to derive constraints on the constitutive behavior of the SMA material. Thus the resulting models do not directly depend on the behavior of the material at the microscopic level, but the effective behavior of the polycrystalline SMA. These models have the advantage of being easily integrated into a structural analysis computational framework, e.g., using the finite element method.

Some of the early three-dimensional models from this group were derived by generalizing one-dimensional results, such as the models by Liang and Rogers (1992), Brinson (1993), Boyd and Lagoudas (1994) and Tanaka et al. (1995). In a publication by Lagoudas et al. (1996) it has been shown that the various phenomenological models can be unified under a common thermomechanical
framework. The differences among the different models arise from the specific choices of the transformation hardening function. More recent phenomenological models have also been presented by Leclercq and Lexcellent (1996), Aurichio et al. (1997), Reisner et al. (1998), Rengarajan et al. (1998), Levitas (1998), Govindjee and Hall (1999) and Rajagopal and Srinivasa (1999). In a recent work Qidwai and Lagoudas (2000b) presented a general thermodynamic framework for phenomenological SMA constitutive models, which for different choice of the transformation function can be tuned to capture different effects of the martensitic phase transformation, such as pressure dependance and volumetric transformation strain. It should be mentioned, however, that the majority of the models are not able to handle the case of detwinning of the martensitic variants. Notable exceptions are the works by Lagoudas and Shu (1999) and Brinson (1993) where the martensitic volume fraction is decomposed into two parts: twinned and detwinned martensite.

In addition to modeling of the development of transformation strains under pseudoelastic or SME conditions, during martensitic phase transformations, several other modeling issues have also been topics of intensive research. One of the important problems recently addressed by the researchers is the behavior of SMAs under cycling loading. During cycling phase transformation a substantial amount of plastic strains is accumulated. In addition, the transformation loop evolves with the number of cycles and TWSME is developed. Based on the experimental observations researchers have attempted to create models able to capture the effects of cycling loading. One-dimensional models for the behavior of SMA wires under cycling loading have been presented by Tanaka et al. (1995), Lexcellent and Bourbon (1996), Lexcellent et al. (2000) and Abeyaratne and Kim (1997), among others. A three-dimensional formulation is given by Fischer et al. (1998). Their model defines a transformation function to account for the development of the martensitic phase transformation and a separate yield function to account for the development of plastic strains. However, neither the identification of the material parameters nor implementation of the model is presented in that work. One of the most recent works on the cyclic behavior of SMA wires has been presented in a series of papers by Bo and Lagoudas (1999a,b,c) and Lagoudas and Bo (1999). In that work most of the issues regarding behavior of SMA wires under cycling loading, including the development of TWSME, have been addressed. The calibration of the model has been performed using experimental data for SMA wires undergoing thermally induced phase transformation under applied load. The dependance of the transformation strain on the level of the applied stress has also been addressed. The modeling results have been compared with experimental data for the case of SMA wire actuators thermally activated under load and have been found to be in good agreement.

1.3. Objectives and outline of the present research effort

As described in the literature review above, the experimental observations for SMAs undergoing cyclic loading in the pseudoelastic regime have shown that a significant part of the developed strain is not recovered upon unloading. This effect has been attributed to the development of plastic strains during the thermomechanical cycling of dense SMAs undergoing phase transformation. Similar effects have also been observed during cyclic mechanical loading of porous SMAs.

The majority of the models found in the literature describing the development of plastic strains in SMAs have one-dimensional formulation. Since the ultimate goal of the current study is to be able to accurately model the behavior of porous SMAs, where three-dimensional effects must be taken into account, successful modeling requires a three-dimensional model for the dense SMA matrix undergoing simultaneous transformation and plasticity. In addition to modeling porous SMAs, a three-dimensional formulation is also needed for modeling various other SMA components, such as torque tubes. In contrast to the earlier work by Bo and Lagoudas (1999a), which considered thermally-induced phase transformation, the current paper will focus on the modeling of stress-induced transformation, where both transformation and
plastic strains occur simultaneously as a result of the applied stress.

Therefore, the main objective of the first part of this two-part work will be to develop a three-dimensional constitutive model for fully dense SMAs which is able to account for simultaneous development of transformation and plastic strains during phase transformation under applied loads and evolution of the material behavior during cyclic loading. The three-dimensional model development will follow the methodology presented for the one-dimensional case by Bo and Lagoudas (1999a,b,c) and Lagoudas and Bo (1999), of transformation induced plasticity. Since the above-mentioned works describe the behavior of SMAs undergoing thermally-induced transformation, the necessary modifications to adapt the formulation for the case of stress-induced martensitic transformation will be made in this work.

The remainder of the paper is organized as follows: in Section 2 the derivations of the fully dense SMA constitutive model are given. The material parameters for the model are identified in Section 3 and a procedure for their estimation is presented. The results of the model and their comparison with the available experimental data are presented in Section 4. A summary of the current research effort and conclusions are presented in Section 5.

The direct notation is adopted in this work. Capital bold Latin letters represent fourth-order tensors (stiffness $\mathbf{S}$, etc.) while bold bold Greek letters are used to denote second-order tensors (stress $\mathbf{\sigma}$, strain $\mathbf{e}$). Regular font is used to denote scalar quantities as well as the components of the tensors. Multiplication of two fourth-order tensors $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{AB} = (AB)_{ijkl} \equiv A_{ijpq}B_{pqkl}$, while the operation ";'" defines contraction of two indices when a fourth-order tensor acts on a second-order one, $\mathbf{A} : \mathbf{E} \equiv A_{ijkl}E_{kl}$.

2. Thermomechanical constitutive modeling of fully dense polycrystalline shape memory alloys

In this section the derivation of a three-dimensional thermomechanical constitutive model for SMAs undergoing cyclic loading which results in simultaneous development of transformation and plastic strains will be presented. The model is an extension of the one-dimensional model presented by Bo and Lagoudas (1999a,b,c) and Lagoudas and Bo (1999) for transformation induced plasticity due to thermally induced phase transformation under loading to three dimensions. Of particular interest in this model is the evolution of plastic strains during stress-induced martensitic phase transformation as well as the non-linear transformation hardening.

The basic ideas for the formulation of the current model have been presented in the literature by many authors, such as Boyd and Lagoudas (1994), Tanaka et al. (1995), Lagoudas et al. (1996), Qidwai and Lagoudas (2000b) and Bo and Lagoudas (1999a,b,c), among others. However, the most important distinctions between the model presented here and the one reported by Bo and Lagoudas (1999a) must be pointed out. The major difference between the two models is that the current model is capable of simulating three-dimensional behavior of SMAs while the model by Bo and Lagoudas (1999a) has only been implemented for the case of one-dimensional SMA wires.

Another difference between the current model and the model by Bo and Lagoudas (1999a) is the way of calibrating the model parameters. While the previous publications (Bo and Lagoudas, 1999a,b,c; Lagoudas and Bo, 1999) have been devoted exclusively to characterizing the behavior of SMA wires undergoing thermally-induced phase transformation and transformation induced plasticity under cyclic thermal loading, the current work is aimed at characterizing the SMAs undergoing stress-induced phase transformation and simultaneous transformation and plasticity evolution with the number of mechanical cycles. Thus, the procedure for estimation of the material parameters, presented in Section 3 utilizes data for SMAs undergoing stress-induced phase transformation. While the present model can still be used to model thermally-induced phase transformation, to obtain accurate results some of the material parameters may need to be re-calibrated. The evolution equation for the plastic strain used here also differs from the expression presented by Bo and Lagoudas (1999b). As experimentally
observed, for the case of cycling under mechanical loading the residual plastic strain saturates after a certain number of cycles. This saturation, however, is not observed for the case of thermally-induced phase transformation under applied load, where the plastic strain continues to accumulate. Therefore, the use of the current expression is motivated by the experimental observations for the evolution of plastic strains during phase transformation under mechanical load.

Finally, a functional form of the back stress different from the one used by Bo and Lagoudas (1999a) is used in the current work. The current functional form of the back stress simplifies the estimation of the material parameter and it is simpler to numerically implement. In addition, evolution equations for the back stress material parameters used here are also different from the ones presented by Bo and Lagoudas (1999b). The evolution of these parameters in the work of Bo and Lagoudas (1999b) depends on the effective accumulated plastic strain for a thermally-induced phase transformation. In the current work the evolution of the back stress parameters depends on the accumulated detwinned martensitic volume fraction. Since the current work involves phase transformation under applied mechanical load, for the case of full transformation the detwinned martensitic volume fraction is proportional to the number of loading/unloading cycles. For the case of thermally-induced transformation, however, the detwinned martensitic volume fraction will also depend on the level of applied stress. Thus, choosing the detwinned martensitic volume fraction as a governing parameter in modeling the evolution of the material parameters during thermally-induced transformation cycling may create difficulties in cases where the applied load changes. In the loading cases considered in the current work, however, the choice is appropriate, since the changes in the material response with the number of cycles can be directly observed during an experiment.

2.1. Experimental observations for polycrystalline SMAs undergoing cyclic loading

As discussed in the introduction, the behavior of SMAs under cyclic loading has been studied by a number of researchers. A set of experimental results\(^1\) presented by Strnadel et al. (1995b) showing the SMA response undergoing cycling stress-induced transformation is shown in Fig. 1. The results shown on the figure are for three different NiTi alloys and the tests have been performed above the austenitic finish temperature. Two different tests were performed: cyclic loading with a constant maximum value of strain and cyclic loading with a constant maximum value of stress. Both sets of the results are shown in the figure. The accumulated plastic strain during loading is shown in Fig. 2.

Several observations can be made from these figures. First, it can be seen that during the cycling loading a substantial amount of unrecoverable plastic strain accumulates. As shown on Fig. 2, the rate of accumulation of plastic strain is high during the initial cycles and asymptotically goes to zero with the increase of the number of cycles, as the plastic strain reaches a saturation value. The second observation is that the value of critical stress for onset of the transformation decreases with the number of cycles. The third observation is the substantial increase of the transformation hardening. In addition, in some cases it is also observed that the value of the maximum transformation strain decreases with the number of cycles. Finally, it can also be seen that the area enclosed by the transformation loop decreases. Similar observations have also been reported by other researchers (see, for example, the works of McCormick and Liu, 1994; Strnadel et al., 1995a; Lim and McDowell, 1994; Kato et al., 1999; Sehtoglu et al., 2001). Thus, the constitutive model presented in this work will address the effects described above, which are common for most polycrystalline NiTi SMAs.

The formulation of the model starts with the definition of Gibbs free energy. The total Gibbs free energy of a polycrystalline SMA (Bo and Lagoudas, 1999a) is given by

\[
G(r, T, \eta; e^t, e^p, n) = C_0 \left( \frac{\sigma}{r} S : r - \frac{1}{2} \sigma : S + \sigma : [e^t(T - T_0) + e^p + e^m] \right) \\
- \frac{1}{\rho} \int_0^\xi \left( \frac{\partial e^t}{\partial \tau} + \eta \right) d\tau \\
- c \left[ (T - T_0) - T \ln \left( \frac{T}{T_0} \right) \right] \\
- s_0(T - T_0) + G^{ch} + G^p. \tag{1}
\]

In the above equation, \( \sigma, e^t, e^p, \xi, T, T_0 \) are the Cauchy stress tensor, transformation strain tensor, plastic strain tensor, martensitic volume fraction,
temperature and reference temperature, respectively. \( S, \bar{\mathbf{a}}, \rho, c \) and \( s_0 \) are the compliance tensor, thermal expansion coefficient tensor, density, specific heat and specific entropy at the reference state, respectively. The above effective material properties are calculated in terms of the martensitic volume fraction \( \xi \) using the rule of mixtures \(^2\) as

\[
\begin{align*}
S &= S^A + \xi (S^M - S^A) = S^A + \xi \Delta S, \\
\bar{\mathbf{a}} &= \bar{\mathbf{a}}^A + \xi (\bar{\mathbf{a}}^M - \bar{\mathbf{a}}^A) = \bar{\mathbf{a}}^A + \xi \Delta \bar{\mathbf{a}}, \\
c &= c^A + \xi (c^M - c^A) = c^A + \xi \Delta c, \\
s_0 &= s_0^A + \xi (s_0^M - s_0^A) = s_0^A + \xi \Delta s_0.
\end{align*}
\]  

\( G^h \) and \( G^p \) are the specific Gibbs chemical free energy at the reference state and the interaction energy induced by plastic strains in the austenitic phase, respectively. Finally, \( \alpha \) and \( \eta \) are the back and drag stresses, which are introduced to describe the influence of local stresses induced by the transformation and plastic strains on the phase transformation.

The second law of thermodynamics is used in the procedure outlined by Coleman and Noll (1963) to derive the following constitutive equations:

\[
\begin{align*}
\varepsilon &= -\rho \frac{\partial G}{\partial \sigma} = S : \sigma + \bar{\mathbf{a}}(T - T_0) + \varepsilon^i + \varepsilon^p, \\
\sigma^v &= -\rho \frac{\partial G}{\partial T} = 1 / \rho \sigma : \bar{\mathbf{a}} + c \ln \left( \frac{T}{T_0} \right) + s_0.
\end{align*}
\]

The local dissipation rate (strong form of the local form of the second law) is given by

\[
-\rho \frac{\partial G}{\partial \xi} = \pi \dot{\xi} \geq 0.
\]

In the above equation \( \pi \) is the thermodynamic force conjugate to \( \xi \) and is given by

\[
\pi = \frac{1}{2} \sigma : \Delta S + \sigma : \Delta \bar{\mathbf{a}}(T - T_0)
\]

\[
+ \sigma : \frac{\partial \varepsilon^i}{\partial \xi} + \alpha : \frac{\partial \varepsilon^i}{\partial \xi} + \eta
\]

\[
- \rho \Delta c \left[ T - T_0 - T \ln \left( \frac{T}{T_0} \right) \right]
\]

\[
+ \rho \Delta s_0 (T - M^{th}) + Y.
\]

The material parameter \( M^{th} \) is introduced in Eq. (6) as a combination of other parameters as

\[
M^{th} = T_0 + \frac{1}{\rho \Delta s_0} (Y + \rho \Delta G^h),
\]

and \( Y \) is a material constant representing a measure of the internal dissipation during phase transformation (Bo and Lagoudas, 1999a). The effective stress \( \sigma^{eff} \) is defined as a sum of the applied stress \( \sigma \) and the back stress \( \alpha \) and is the thermodynamic force conjugate to transformation strain \( \varepsilon^t \):

\[
\sigma^{eff} = -\rho \frac{\partial G}{\partial \varepsilon^t} = \sigma + \alpha.
\]

2.3. Evolution of internal state variables

The evolution equations of internal state variables \( \dot{\xi}, \dot{\varepsilon^i}, \dot{\varepsilon^p}, \alpha \) and \( \eta \) are presented here. As stated earlier, the transformation strain, plastic strain and back and drag stresses evolve with the martensitic volume fraction \( \xi \). Therefore, the evolution of \( \dot{\xi} \) is discussed first.

2.3.1. Martensitic volume fraction

To obtain an evolution equation for \( \dot{\xi} \), a standard formalism of thermodynamic dissipation potentials (see, for example, Edelen, 1974) is utilized. Introducing a dissipation potential \( \phi(\pi; \sigma, T, \xi) \) the evolution equation for the internal state variable \( \xi \) is given by

\[
\dot{\xi} = \lambda \frac{\partial \phi(\pi; \sigma, T, \xi)}{\partial \pi},
\]
and \( \lambda \) satisfies the following Kuhn–Tucker conditions:

\[
\lambda \geq 0, \quad \phi \leq Y^*, \quad \lambda (\phi - Y^*) = 0. \tag{10}
\]

The quantity \( Y^* \) related to the dissipation rate of the system is assumed to be constant during phase transformation. Assuming a convex quadratic functional representation of the dissipation potential

\[
\phi = \frac{1}{2} \pi^2, \tag{11}
\]

and using the Kuhn–Tucker conditions (10) for the case of \( \lambda > 0 \) the phase transformation condition is given by

\[
\pi = \pm \sqrt{2Y^*} \equiv \pm Y. \tag{12}
\]

The application of the thermodynamical constraint (5) leads to the following choice of the transformation function \( \Phi \):

\[
\Phi = \begin{cases} 
\pi - Y, & \dot{\zeta} > 0, \\
-\pi - Y, & \dot{\zeta} < 0.
\end{cases} \tag{13}
\]

Constraints on the evolution of \( \zeta \) are expressed in terms of the Kuhn–Tucker conditions as

\[
\dot{\zeta} \geq 0, \quad \Phi \leq 0, \quad \Phi \dot{\zeta} = 0,
\]

\[
\dot{\zeta} \leq 0, \quad \Phi \leq 0, \quad \Phi \dot{\zeta} = 0. \tag{14}
\]

The role of the transformation function \( \Phi \) defined by Eq. (13) is similar to the role of the yield function in theory of rate-independent plasticity. It defines the elastic domain where no phase transformation occurs. The inequality conditions on \( \Phi \) are usually called the consistency conditions and act as a constraint on the admissibility of the state variables. Conditions expressed by Eq. (14) should be satisfied along any loading path. When \( \Phi < 0 \) Eq. (14) enforces the condition \( \dot{\zeta} = 0 \) and the material response is elastic. When \( \Phi = 0 \) the material transforms: forward phase transformation is obtained for \( \dot{\zeta} > 0 \) and reverse for \( \dot{\zeta} < 0 \).

### 2.3.2. Transformation strain

During the martensitic phase transformation the high-symmetry austenitic parent phase transforms into lower-symmetry martensitic phase. Thus, during forward transformation the parent phase can deform into many possible variants. It has been generally assumed that the direction of transformation is determined by the effective stress at each material point. During reverse phase transformation the high-symmetry austenitic phase, consisting of only one variant is recovered. In addition to the direction of the transformation a second factor, representing its magnitude must be taken into account. Differences exist between stress- and thermally-induced phase transformation, which affects the magnitude of the transformation strain.

During the stress-induced phase transformation, simultaneous transformation and reorientation (detwinning) occur in the material. Due to the constraints between different grains in a polycrystalline SMA, different grains transform in a different way. Thus, different martensitic variants are created in different grains and the maximum transformation strain observed for single-crystal SMAs can not be fully achieved. It can be approximately assumed, however, that the martensitic variants are equally detwinned through the stress-induced phase transformation. Therefore, the magnitude of the transformation strain can be characterized by a single material parameter, the maximum transformation strain.

During thermally-induced phase transformation at a constant applied load the achieved transformation strain strongly depends on the value of the load. If an untrained SMA is cooled below the martensitic start temperature without applied load, the resulting martensite will be self-accommodated and no macroscopic transformation strain will be observed. When mechanical load is applied during the cooling, the phase transformation will produce detwinned martensite and a macroscopic transformation strain will be observed. Due to the complex nature of the martensitic phase transformation and the interactions between the different grains in a polycrystalline SMA the degree of “detwinning” will strongly depend on the value of the applied load.

In addition, the transformation strain depends on the conditioning of the material. It has been experimentally observed that the training achieved by cycling of the SMA material has a very strong effect on the relationship between the value of the transformation strain and the applied load. For a
trained SMA it is possible to achieve non-zero transformation strain even when the applied load is zero. In fact, to fully suppress the transformation strain a load in the direction opposite to the training load must be applied, as shown by Stalmans et al. (1992). Therefore, the magnitude of the transformation strain for the case of thermally-induced phase transformation depends on the applied load as well as on the previous thermo-mechanical loading.

During the reverse phase transformation it is assumed that the transformation strain will decrease with the decrease of $\xi$ from its maximum value to zero. Based on the above discussion the following evolution equation for the transformation strain $\xi$, proposed by Bo and Lagoudas (1999a) and similar to the one used by Lagoudas et al. (1996) is used in this work:

$$\dot{\xi} = \Lambda \xi,$$  \hspace{1cm} (15)

where $\Lambda$ is the transformation direction tensor and is given by

$$\Lambda = \begin{cases} \frac{3}{2} H^{\text{cur}} \frac{\sigma^{\text{eff}}}{\sigma^{\text{eff}}}, & \xi > 0, \\ \frac{\xi_{\text{max}}}{\xi_{\text{max}}}, & \xi < 0. \end{cases}$$  \hspace{1cm} (16)

The quantity $H^{\text{cur}}$ appearing in the above equation is defined to be the maximum current transformation strain and is a function of the applied stress. It is a measure of the degree of "detwinning" of the martensitic variants. $\xi_{\text{max}}$ and $\xi_{\text{max}}$ are defined to be the transformation strain and the martensitic volume fraction at the beginning of the reverse phase transformation, respectively. The deviatoric part of the effective stress $\sigma^{\text{eff}}$ and the effective von Mises stress $\bar{\sigma}^{\text{eff}}$ are defined as

$$\sigma^{\text{eff}} = \sigma - \frac{1}{3}(\text{tr} \sigma) 1, \hspace{1cm} \bar{\sigma}^{\text{eff}} = \sqrt{\frac{3}{2} \| \sigma^{\text{eff}} \|},$$  \hspace{1cm} (17)

where $\| \cdot \|$ is the inner product of the quantity and $1$ is the rank two identity tensor. The condition for evaluation of $H^{\text{cur}}$ used here is the one suggested by Lagoudas and Bo (1999), i.e., $H^{\text{cur}}$ is evaluated from the condition that the effective applied stress $\bar{\sigma}$ is equal to the effective back stress $\bar{\sigma}$ at the value of the martensitic volume fraction $\xi = 1$:

$$\bar{\sigma} = \bar{\sigma} |_{\xi=1}.$$  \hspace{1cm} (18)

As shown in Section 2.3.4, the back stress $\bar{\sigma}$ is a function of both $\xi$ and $H^{\text{cur}}$. Thus, for $\xi = 1$, (18) becomes an equation of one variable only, $H^{\text{cur}}$ and it can be solved to determine its value. This also implies that when the value of stress changes, a new value of the current maximum transformation strain should be calculated. The latter case arises not only during stress-induced phase transformation, but also during thermally-induced phase transformation when the material is constrained, e.g., in applications where SMA acts as an actuator.

It should be mentioned here that the current three-dimensional formulation of the model will not properly take into account the development of two-way shape memory effect in its full generality, but only in special cases. The one-dimensional reduced model, however, will be able to account for the TWSME. This limitation of the model is caused by the fact that the current choice for calculating $H^{\text{cur}}$ cannot take into account the direction of the developed TWSME. To be able to properly model the training and development of TWSME a tensorial quantity must be introduced, which will replace $H^{\text{cur}}$. Note, however, that for the case of stress-induced martensitic phase transformation the current formulation is still suitable. This is due to the fact that the transformation strain during stress-induced phase transformation will develop in the direction of the applied stress.

### 2.3.3. Plastic strain

The plastic strain considered here is different from conventional plasticity in metals. The observable macroscopic plastic strain is developed simultaneously with the transformation strain during martensitic phase transformation. It is a collective result of the accommodation of different martensitic variants during the phase transformation. Due to the misfit between the austenite–martensite interfaces significant distortion is created. In addition, in a polycrystalline SMA different grains transform in a different manner. This causes additional distortion at the grain boundaries. These two phenomena act in concert and the
final result is an observable macroscopic plastic strain. It should be noted that this plastic strain occurs at stress levels much lower than the plastic yield limit of the material without phase transformation (Saburi, 1999).

Similar to the evolution of transformation strain, the direction of plastic strain is determined by the direction of the applied stress. In addition to the applied stress another factor must be taken into account. Lim and McDowell (1994) have suggested that the plastic strain rate depends on the magnitude of $\dot{\xi}$. In this work, however, a different approach, outlined by Bo and Lagoudas (1999b) will be used. It is assumed that the self-accommodating martensitic phase transformation does not result in plastic strain development. Therefore, the plastic development of the plastic strain is connected to the detwinned martensitic volume fraction $\xi^d$. It is seen from its definition that $\xi^d$ indicates that the plastic strain reaches a saturation value. Thus, to be able to accurately model stress-induced phase transformation the following form of $\xi^d$ is introduced as:

$$\xi^d = \int_0^t |\dot{\xi}^d(\tau)| \, d\tau$$

and the following evolution equation for $\xi^d$ is proposed

$$\dot{\xi}^d = \Lambda^p \xi^d.$$  

It is seen from its definition that $\dot{\xi}^d = |\dot{\xi}^d|$. The quantity $\Lambda^p$ is the plastic direction tensor and its functional form is discussed next.

One form of $\Lambda^p$ is suggested by Bo and Lagoudas (1999b) for the one-dimensional case. It depends on the value of the applied stress, the accumulated detwinned martensitic volume fraction and on the value of the plastic strain itself. As explained by Bo and Lagoudas (1999b) the plastic strain predicted by their model never reaches a saturation value. While this is observed during temperature-induced phase transformation, for the case of stress-induced transformation the experimental results reported by Strnad et al. (1995b) indicate that the plastic strain reaches a saturation value. Thus, to be able to accurately model stress-induced phase transformation the following form of $\Lambda^p$ is used:

$$\Lambda^p = \begin{cases} \frac{3}{2} C_1^p \sigma^\text{eff} \ e^{\frac{-d^p}{C_2^p}}, & \dot{\xi} > 0, \\ C_1^e \frac{\xi^d_{\text{max}}}{\bar{\xi}^d_{\text{max}}} \ e^{\frac{-d^e}{C_2^e}}, & \dot{\xi} < 0. \end{cases}$$  

It can be seen from Eq. (21) that the rates of the transformation strain and the plastic strain during phase transformation are proportional. An evolution equation for the plastic strain during stress-induced phase transformation, similar to Eq. (21) has also been proposed by Lim and McDowell (1994). However, in their equation the governing parameter is the accumulated martensitic volume fraction, while in the equation proposed in this work the governing parameters is the accumulated detwinned martensitic volume fraction. In addition, the equation proposed here is valid for three-dimensional case, while the equation presented by Lim and McDowell (1994) is given only for the one-dimensional case. This form of the plastic direction tensor enables a saturation of the plastic strain after a certain number of cycles. The material parameters $C_1^p$ and $C_2^p$ govern the saturation value as well as the number of cycles necessary for the plastic strain to saturate. The quantity $\bar{\xi}^d_{\text{max}}$ is the effective transformation strain and is defined as

$$\bar{\xi}^d_{\text{max}} = \sqrt{\frac{2}{3} \| \xi^d_{\text{max}} \|}.$$
equations in terms of the martensitic volume fraction. Since the back stress is a tensorial quantity, its functional form also contains the direction of the accumulated transformation strain. Alternatively, an evolution equation can also be specified which will have a form similar to the evolution equation for the transformation strain (see Eqs. (15) and (16)).

In this work the back stress is assumed to have the following polynomial functional representation:

$$\mathbf{x} = -\gamma \sum_{i=1}^{N^b} D^b_i (H_{\text{out}} \xi)^{(i)},$$

(23)

where $N^b$ is the degree of the polynomial and $D^b_i$ are the coefficients associated with the back stress. Note that the form of the back stress used here differs from the expression used by Bo and Lagoudas (1999b), where a logarithmic function has been used. The use of a polynomial expression significantly simplifies the estimation of the material parameters and the calibration of the model. Using Eq. (23) the back stress parameters can be calibrated using a least square fit of the experimental data, while the logarithmic expression used by Bo and Lagoudas (1999b) would result in a non-linear optimization problem.

The expression for $\eta$ used in this work is similar to the one used by Bo and Lagoudas (1999b):

$$\eta = -D^d_1 [-\ln(1 - \xi)]^m + D^d_2 \xi,$$

(24)

where $D^d_1, D^d_2$ and $m_1$ are parameters governing the evolution of the drag stress.

Here it must be mentioned that two time scales are recognized: the fast time scale, which is connected with the phase transformation, and the slow time scale, connected with the evolution of the material response during cyclic loading. It should also be noted that the back and drag stress parameters as well as the other material parameters (e.g., $Y$, $M^0$) do not remain constant during cyclic loading but are functionals of the martensitic volume fraction $\xi$ and evolve during cyclic loading (i.e., with the slow time scale). While the phenomena connected with the fast time scale were described in this section, the slow time scale effects are described next.

### 2.4. Evolution of the hysteretic response of an SMA undergoing mechanical cyclic loading

During mechanical cyclic loading there are several characteristic changes of the thermomechanical response of SMAs. Along with the accumulation of non-recoverable plastic strain, changes of the hysteresis loop have also been experimentally observed (Lim and McDowell, 1994; Strnad et al., 1995). The hysteresis loop progressively evolves with the number of cycles, until a stabilization point is reached. Some of the characteristic changes of the hysteresis loop are: (i) decrease of the stress level necessary for the onset of the transformation; (ii) increase of the transformation hardening; (iii) decrease of the width of the hysteresis loop; (iv) decrease of the maximum transformation strain. In addition, as noted in the literature (McCormick and Liu, 1994; Bo and Lagoudas, 1999b) the martensitic start temperature at zero applied stress $M^0_0$ can also change during the transformation cycling.

The accumulation of the plastic strain has been addressed in the previous section, where an evolution equation for $\varepsilon^p$ has been proposed. This section addresses the evolution of the hysteresis loop. The approach taken here is to identify two sets of parameters—the first set for the material which has not undergone any thermodynamic loading and the second set for the material which has undergone transformation cycling and the hysteresis loop has stabilized. Then, having identified these two sets of material parameters, evolution equations are proposed such that during the cycling the material parameters evolve from the first set to the second set. The procedure is described in detail in a sequel.

First, the evolution of the back stress parameters $D^b_i$ is prescribed. The initial and final values of the back stress parameters are denoted by $(D^b_i)^{\text{init}}$ and $(D^b_i)^{\text{fin}}$. Following the work of Bo and Lagoudas (1999b) it is assumed that the evolution of the parameters $D^b_i$ is governed by the same equation, which is selected to be

$$D^b_i = (D^b_i)^{\text{fin}} + (D^b_i)^{\text{init}} - (D^b_i)^{\text{fin}} e^{-\lambda_i \xi}.$$

(25)

As seen from Eq. (25) it is assumed that the back stress parameters change with the evolution of the
accumulated detwinned martensitic volume fraction $\zeta^d$. In their work Bo and Lagoudas (1999b) have assumed that the evolution of $D^b$ is governed by the change in plastic strain. However, as indicated by Eqs. (20) and (21) the plastic strain $\varepsilon^p$ and $\zeta^d$ are connected. Choosing $\zeta^d$ to be the governing parameter for the change of $D^b$ simplifies the model calibration, since for stress-induced transformation where full detwinning takes place $\zeta^d$ is proportional to the number of cycles, i.e., $\zeta^d = 2N$, where $N$ is the number of cycles. The parameter $\lambda_1$ in Eq. (25) is a positive material constant which governs the increasing rate of $D^b$.

As explained in Section 2.3.2 the current maximum transformation strain $H^\text{cur}$ is calculated using the effective back stress $\tilde{\sigma}$. Since the maximum transformation strain $H$ is a limit value of $H^\text{cur}$ then it is assumed that the change of $H$ obeys the same governing equation as the change of $D^b$. Therefore, $H$ is given by

$$H = H^\text{fin} + (H^\text{init} - H^\text{fin})e^{-\lambda_1\zeta^d}. \quad (26)$$

Similar evolution equations are proposed for the drag stress parameters $D^d$. The evolution of $D^d$ is described by

$$D^d = (D^d)^\text{fin} + ((D^d)^\text{init} - (D^d)^\text{fin})e^{-\lambda_2\zeta}. \quad (27)$$

where $\lambda_2$ is a material parameter governing the evolution of $D^d$. As seen from Eq. (27) the evolution of the drag stress parameters is governed by the total accumulated martensitic volume fraction $\zeta$ and not its detwinned portion $\zeta^d$. This is related to the fact that microstructural changes can be induced by cyclic self-accommodating phase transformation (Bo and Lagoudas, 1999b). It should be noted that for stress-induced phase transformation with large values of the applied stress equations (25) and (27) are identical (if, of course, $\lambda_1 = \lambda_2$), since in this case $\zeta = \zeta^d$.

Finally, the evolution of the material parameters $Y$, $M^\text{th}$ and $\rho \Delta s_0$ is considered. The equations governing the change of these parameters are similar to Eq. (27):

$$Y = Y^\text{fin} + (Y^\text{init} - Y^\text{fin})e^{-\lambda_2\zeta}, \quad (28)$$

$$M^\text{th} = (M^\text{th})^\text{fin} + ((M^\text{th})^\text{init} - (M^\text{th})^\text{fin})e^{-\lambda_2\zeta}, \quad (29)$$

$$\rho \Delta s_0 = (\rho \Delta s_0)^\text{fin} + ((\rho \Delta s_0)^\text{init} - (\rho \Delta s_0)^\text{fin})e^{-\lambda_2\zeta}. \quad (30)$$

The value for the parameter $\lambda_1$ can be obtained by performing cyclic loading during which the material undergoes stress-induced transformation, while the value of $\lambda_2$ can be determined by performing thermal cycling with no applied stress. While the evolution equation for the back stress parameters (Eqs. (25) and (26)) involves the accumulated detwinned martensitic volume fraction $\zeta^d$, the other evolution equations (27)–(30) involve the accumulated total martensitic volume fraction $\zeta$. In the case of self-accommodated thermally induced transformation cycling the value of $\zeta^d$ is zero. Then, any change of the material parameters will be caused by the change of $\zeta$, which will allow the determination of the value of $\lambda_2$ from any of Eqs. (27)–(30), assuming that $\lambda_2$ remains the same for all evolution equations (27)–(30).

As discussed above, two sets of the material parameters need to be identified—the initial set, characterizing the initial response of the annealed material and the final set, characterizing the stable material response. Then, having identified these two sets, the material parameters continuously change according to the evolution equations. However, this situation poses a problem in identifying the initial and final values of the parameters. Indeed, if the material parameters change continuously during the identification of the first set, it would be extremely difficult to take into account the change during the first cycle due to the non-linearity introduced by that change.

This problem is addressed by keeping the value of the material parameters constant during forward or reverse phase transformation. The parameters will be recalculated according to the evolution equations when a reversal of the phase transformation occurs. Thus, the change in sign of the martensitic volume fraction rate $\dot{\zeta}$ triggers the change of the material parameters. Note, however, that the above procedure is applied only for the material parameters. The plastic strain during cyclic loading is continuously calculated during both forward and reverse phase transformation, according to the evolution equation (20).
2.5. Modeling of minor hysteresis loops

An important part of the thermomechanical constitutive modeling of SMAs is the accounting for the minor hysteresis loops. In the context of the presented model a major loop is characterized by a full transformation cycle with the martensitic volume fraction $\xi$ monotonically increasing from 0 to 1 and then monotonically decreasing from 1 to 0. On the other hand, during a minor loop the martensitic volume fraction $\xi$ initial value is strictly greater than 0 and less than 1.

The approach to modeling the minor hysteresis loops used in this work is the one presented by Bo and Lagoudas (1999c). The main idea behind the modeling of minor loops is the modification of the transformation function, depending on whether the loading path follows a major or a minor loop. The details of the approach can be found in the work of Bo and Lagoudas (1999c). Here it will only be mentioned that an additional parameter $c$ controlling the shape of the minor loops is introduced.

3. Estimation of material parameters

As a final step to characterizing the thermomechanical behavior of SMAs undergoing cyclic loading, in this section the determination of the material parameters entering the model is discussed. Three groups of material parameters are identified. First, the parameters which are necessary to describe a stable transformation cycle are determined. In the context of the current model the stable transformation cycle is defined as a thermomechanical loading cycle during which no plastic strains are developed and the material parameters remain constant. Note that some of these material parameters, such as the elastic moduli, thermal expansion coefficients, etc., are material constants and do not change during transformation cycling, while others, such as the back stress parameters, drag stress parameters, etc., are material functions and evolve with transformation cycling.

The second group includes the material parameters which describe the behavior of SMAs under cycling loading. These are the material parameters governing the evolution of plastic strains as well as the evolution of the material functions from the first group.

Finally, the third group encompasses the parameters governing the SMA behavior during minor hysteresis loops. The material parameters will be determined using uniaxial tests. Therefore, the one-dimensional reduction of the model is given next.

3.1. One-dimensional reduction of the model

During uniaxial loading in $x_1$ direction the stress tensor has one non-zero component. Thus, the components of the stress tensor are given by:

$$
\sigma_{11} = \sigma \neq 0, \quad \sigma_{ij} = 0 \quad \text{for all other } i, j.
$$

(31)

In the equation above $\sigma$ is the applied uniaxial stress. The transformation and plastic strain components are given by

$$
e_{11} = e^t, \quad e_{22} = e_{33} = -\frac{1}{2}e^t, \quad e_{ij} = 0
$$

for all other $i, j$.

(32)

$$
e_{11}^p = e^p, \quad e_{22}^p = e_{33}^p = -\frac{1}{2}e^p, \quad e_{ij}^p = 0
$$

for all other $i, j$.

(33)

where $e^t$ and $e^p$ are the uniaxial transformation and plastic strains, respectively, assuming that both result in isochoric deformations. Assuming isotropic elastic properties, the constitutive equations (3) in the one-dimensional form become

$$
e_{11} = \varepsilon = S\sigma + \tilde{a}(T - T_0) + e^t + e^p,
$$

(34)

where $\varepsilon$ is the uniaxial total strain, $\tilde{a}$ is the thermal expansion coefficient and $S$ is the elastic compliance given by $S = 1/E$, where $E$ is the Young’s elastic modulus.

The evolution equations for the transformation and plastic strain become

$$
\dot{e}^t = A\dot{\xi},
$$

(35)

$$
\dot{e}^p = A^p\dot{\xi}^d.
$$

(36)

The uniaxial components of the transformation and plastic direction tensors are given by

\^
The transformation function is given by

\[ A = A_{11} = \begin{cases} H^\text{cur} & \dot{\xi} > 0, \\ \frac{\varepsilon_{\text{max}}^1}{\dot{\varepsilon}_{\text{max}}}, & \dot{\xi} < 0, \end{cases} \]

\[ A^p = A_{11}^p = \begin{cases} C^p & \dot{\xi} > 0, \\ \frac{\varepsilon_{\text{max}}^1}{\dot{\varepsilon}_{\text{max}}}, & \dot{\xi} < 0. \end{cases} \]

The uniaxial effective stress \( \sigma_{\text{eff}} \) is defined in terms of the applied stress \( \sigma \) and the back stress \( \lambda \) as

\[ \sigma_{\text{eff}} = \sigma + \lambda, \quad \lambda = -\frac{3}{2} \varepsilon^I \sum_{i=1}^{3} D^I_i (H^\text{cur} \dot{\xi})^I. \]

The expression for the drag stress \( \eta \) is the same as given by Eq. (24)

\[ \eta = D^I_1 [-\ln(1 - \xi)]^I + D^I_2 \xi; \]

The transformation function is given by

\[ \Phi = \begin{cases} \pi - Y = 0, & \dot{\xi} > 0, \\ -\pi - Y = 0, & \dot{\xi} < 0, \end{cases} \]

where the thermodynamic driving force \( \pi \) is given by

\[ \pi = \frac{1}{2} \sigma^2 \Delta S + \sigma \Delta \tilde{\zeta}(T - T_0) + \sigma_{\text{eff}} A + \eta \\
- \rho \Delta c \left[ T - T_0 - T \ln \left( \frac{t}{T_0} \right) \right] \\
+ \rho \Delta s_0 (T - M^{\text{br}}) + Y. \]

The quantity \( \Delta S \) is the difference between the elastic compliances of the austenitic and martensitic phases and is given by

\[ \Delta S = \frac{1}{E^M} - \frac{1}{E^A} \]

and \( \Delta \tilde{\zeta} \) is the difference between the thermal expansion coefficients of the austenite and martensite.

The uniaxial tangent stiffness is given by

\[ L' = E - \frac{\chi E^2 \partial_\sigma \Phi}{\partial E \partial_\sigma \Phi - \partial_\zeta \Phi}, \]

where \( \partial_\sigma \Phi \) and \( \partial_\zeta \Phi \) are the derivatives of the transformation function with respect to stress and the martensitic volume fraction, respectively, and the quantity \( \chi \) is given by

\[ \chi = \sigma \Delta S + \Delta \tilde{\zeta}(T - T_0) + A + \text{sign}(\dot{\xi}) \frac{H^\text{cur}}{H} A^p. \]

The determination of the material parameters of all of the three groups is described in detail in the following sections. The tests which are necessary to be performed will be discussed and a parametric study for selected material parameters will be performed. The material parameters for NiTi reported by Bo et al. (1999) and shown in Table 1 will be used during the parametric study. The parameters shown in Table 1 are obtained for NiTi wires undergoing thermally-induced phase transformation and may differ from the parameters obtained for SMA undergoing stress-induced phase transformation. However, these material parameters will only be used to carry out parametric studies.

3.2. Material parameters for a stable transformation cycle

This group of material parameters include the elastic compliance tensors of both austenitic and martensitic phases, \( S^A \) and \( S^M \), respectively, their thermal expansion coefficient tensors, \( \alpha^A \) and \( \alpha^M \),

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Material parameters for NiTi SMA characterizing a stable transformation cycle (Bo et al., 1999)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Physical constants</strong></td>
<td><strong>Parameters characterizing the phase transformation</strong></td>
</tr>
<tr>
<td>( E^A = 70.0 \text{ GPa} )</td>
<td>( M^{\text{br}} = 311.0 \text{ K} )</td>
</tr>
<tr>
<td>( E^M = 30.0 \text{ GPa} )</td>
<td>( H = 0.069 )</td>
</tr>
<tr>
<td>( \rho_c^A = 2.12 \text{ MJ/(m}^3\text{ K}) )</td>
<td>( \rho_c^M = 2.12 \text{ MJ/(m}^3\text{ K}) )</td>
</tr>
<tr>
<td>( \chi^A = 11.0 \times 10^{-6} \text{ K}^{-1} )</td>
<td>( \chi^M = 6.6 \times 10^{-6} \text{ K}^{-1} )</td>
</tr>
<tr>
<td>( \rho \Delta s_0 = -0.422 \text{ MJ/(m}^3\text{ K}) )</td>
<td>( \rho \Delta s_0 = -0.422 \text{ MJ/(m}^3\text{ K}) )</td>
</tr>
<tr>
<td>( H = 0.069 )</td>
<td>( Y = 6.0 \text{ MJ/m}^3 )</td>
</tr>
<tr>
<td>( D^1_1 = 3.40 \times 10^3 \text{ MPa} )</td>
<td>( D^1_1 = 8.0 \text{ MPa} )</td>
</tr>
<tr>
<td>( D^2_1 = -2.23 \times 10^3 \text{ MPa} )</td>
<td>( D^2_1 = 1.7 \text{ MPa} )</td>
</tr>
<tr>
<td>( D^3_1 = 8.32 \times 10^3 \text{ MPa} )</td>
<td>( m_1 = 3.5 )</td>
</tr>
<tr>
<td>( D^3_2 = -1.50 \times 10^3 \text{ MPa} )</td>
<td>( D^3_2 = 9.0 \times 10^3 \text{ MPa} )</td>
</tr>
<tr>
<td>( D^3_3 = 1.03 \times 10^3 \text{ MPa} )</td>
<td>( \gamma = 3.0 )</td>
</tr>
</tbody>
</table>
and the specific heat, $c^A$ and $c^M$. The martensitic start temperature $M_0^S$, maximum transformation strain $H$, the difference between the specific entropy per unit volume $\rho \Delta S_0$, the material parameter $Y$, which provides a measure of the internal dissipation during phase transformation, and the parameters associated with the back and drag stresses are also included in this group of parameters.

To determine the elastic compliance tensors $S^A$ and $S^M$ it is assumed that the SMA material behaves isotropically at macroscale. This assumption is reasonable because of the random orientation of the grains in a polycrystalline SMA. Therefore, it is only necessary to find the Young’s elastic moduli and Poisson’s ratios $E^A$, $E^M$, $\nu^A$ and $\nu^M$ of both phases. To determine the elastic constants a uniaxial pseudoelastic test must be performed. The elastic stiffness $E^A$ is determined by calculating the initial slope of the stress–strain curve for a uniaxial pseudoelastic test as shown in Fig. 3. The elastic stiffness of the martensite phase $E^M$ is given by the slope of the stress–strain curve at the point of initial unloading (see Fig. 3). It has been generally assumed in the literature that the Poisson’s ratios of austenite and martensite are equal and a typical value of $\nu^A = \nu^M = 0.33$ has been reported (Lagoudas et al., 1996; Lagoudas and Bo, 1999).

The thermal expansion coefficient tensors $\alpha^A$ and $\alpha^M$ are fully represented by two scalar constants $\alpha^A$ and $\alpha^M$. These constants can be estimated by measuring the slope of strain–temperature curve under constant stress at high temperature for the austenitic phase and low temperature for the martensitic phase. The specific heat constants $c^A$ and $c^M$ can be obtained from a calorimetric test. Note that these constants ($\alpha^A$, $\alpha^M$, $c^A$, $c^M$) are not needed to model stress-induced phase transformation under constant temperature, but they become important when temperature changes.

The martensitic start temperature $M_0^S$ can be obtained from a Differential Scanning Calorimetry (DSC) test. Detailed description of the DSC setup and test is given by Bo et al. (1999). The maximum transformation strain $H$ is obtained by performing a pseudoelastic uniaxial test. Its value is estimated by extending the unloading part of the stress–strain curve using the elastic stiffness of the martensitic phase $E^M$, as shown in Fig. 3.

It should be mentioned that during the initial loading cycles observable plastic strain can develop. One proposed way to separate the plastic strain from the transformation strain is to subtract a portion of the residual plastic strain from the measured value of the transformation strain. Since the plastic strain develops during both forward and reverse transformation, the subtracted value must be adjusted to take into account the plastic strain developed during the reverse phase transformation. As experimentally observed and reflected in the evolution equation for the plastic strain, initially the rate of accumulation of the plastic strain can be approximated by a linear function. Therefore, it is reasonable to assume that the amount of the plastic strain developed during the forward phase transformation is equal to one-half of the total residual plastic strain after one cycle. Thus, in presence of plastic strains, the value of the maximum transformation strain $H$ is obtained by extending the unloading part of the stress–strain curve and subtracting one-half of the total residual strain from the obtained number.

The material parameter $\rho \Delta s_0$ can also be obtained from the pseudoelastic stress–strain curve, schematically shown in Fig. 3. Eqs. (41) and (42) for the transformation function $\Phi$ at the onset of phase transformation ($\xi = 0$) lead to

\[
\rho \Delta s_0 = -\frac{\frac{1}{2}(\sigma_{Ms})^2 \Delta S + \sigma_{Ms} H_{cur}}{T - T_0^b}. \tag{46}
\]
Note that the value of $H_{\text{cur}}$ used in Eq. (46) should be calculated for the corresponding value of stress, which in this case is equal to $\sigma^{\text{Ms}}$. A slightly different procedure for determination of $M_{\text{obs}}^i$ and $\rho \Delta s_0$ is reported by Bo et al. (1999). In their work, the quantity $\rho \Delta s_0$ is calculated from the DSC test and is related to the latent heat released during forward phase transformation and absorbed during reverse phase transformation. The martensitic start temperature $M_{\text{obs}}^i$ is calculated in their work using strain–temperature curve obtained during an isobaric experiment.

To demonstrate the effect of the value of $\rho \Delta s_0$ on the stress–strain response, loading cases with different numerical values of $\rho \Delta s_0$ are simulated. The results of the parametric study are shown in Fig. 4 where only the loading part is shown. It can be seen that with increasing magnitude of $\rho \Delta s_0$ the value of stress for the onset of the transformation also increases. Therefore, it can be seen that the parameter $\rho \Delta s_0$ is connected to the slope of the transformation line on the stress–temperature phase diagram.

The material parameter $Y$ can also be calculated using a pseudoelastic stress–strain curve. The value of $Y$ is related to the total area $A$ enclosed by the hysteresis curve during a complete phase transformation as

$$A = 2Y.$$ (47)
After the value of $H_{\text{cur}}$ reaches the value of the maximum transformation strain $H$, it is assumed that further increase of the stress does not yield further increase of $H_{\text{cur}}$. After that point the value of $H_{\text{cur}}$ is considered to be constant and equal to $H$.

Note that sometimes it is difficult to obtain experimental curves for $\sigma$ versus $H_{\text{cur}}$. In case that this data is unavailable, a typical curve, similar to the one shown in Fig. 6 can be used. The results shown in Fig. 6 have been reported by Lagoudas and Bo (1999) and similar results have also been observed by other researchers. Since for a stress-induced martensite the values of stress are expected to be high enough to induce complete detwinning of the martensitic variants, the errors introduced by this approximation are expected to be relatively small. For temperature-induced phase transformation, however, this experimental data is critical for the accuracy of the model, since small variations in stress may cause large variations of $H_{\text{cur}}$.

To estimate the material parameters entering the expression for the drag stress $\eta$ the tangent stiffness during an isothermal uniaxial pseudoelastic test is used. Assuming that all of the material parameters, except the drag stress parameters, are known, the tangent stiffness $L'$ given by Eq. (44) becomes a function of the drag stress parameters $D_{d1}^d$, $D_{d2}^d$ and $m_1$ and the martensitic volume fraction $\xi$:

$$L'(\xi; D_{d1}^d, D_{d2}^d, m_1) = \frac{E}{\text{DEN}} \left\{ -\frac{E}{(\sigma \Delta S + H_{\text{cur}})^2} \right\},$$

where the denominator is given by

$$\text{DEN} = \frac{E(\sigma \Delta S + H_{\text{cur}})^2}{\sum_{i=1}^{N} iD_i^b (H_{\text{cur}}^b)^{(i-1)}} + \frac{1}{m_1(1-\xi)} D_1^d \left[ -\ln(1-\xi) \right] - D_2^d.$$  

The experimental values of the uniaxial tangent stiffness are obtained by performing a pseudo-elastic test. Thus, having obtained the experimental values of the tangent stiffness, a least-square fit of Eq. (49) is performed to obtain the drag stress parameters $D_{d1}^d$, $D_{d2}^d$ and $m_1$. Note, that the internal parameter $\xi$ cannot be directly measured. Based on numerous numerical experiments performed using the current model it has been found that it is reasonable to assume a linear relationship between the rate of total strain and the rate of the martensitic volume fraction during transformation.

To illustrate the effect of the drag stress parameters on the stress–strain results, a parameter study for different values of $D_{d1}^d$ and $D_{d2}^d$ has
been performed. The loading part of the pseudo-
elastic stress–strain curve for different values of $D_{d1}$ is shown in Fig. 7. It can be seen from the figure that the transformation hardening increases with the increase of the value of $D_{d1}$. In addition, the value of $D_{d1}$ has a very strong effect on the initial transformation hardening. The effect of the parameter $D_{d2}$ on the stress–strain results is shown in Fig. 8. It can be seen that higher value of the parameter $D_{d2}$ leads to smaller slope of the stress–strain curve. This parameter, however, has very little effect on the initial transformation hardening.

The effect of $D_{d2}$ can be explained by comparing the expression for the drag stress in the current model to the one given by Lagoudas et al. (1996) for a model with a polynomial hardening function. If the parameter $D_{d1}$ is taken to be zero, then the parameter $D_{d2}$ corresponds to the linear hardening parameter in the model by Lagoudas et al. (1996). Therefore, it has the same effect for all values of $\xi$ and its value equally affects the slope of the stress–strain curve during the phase transformation.

### 3.3. Material parameters for cyclic loading

This group of parameters includes the parameters $C_{p1}$ and $C_{p2}$, entering the expression for $A^p$ (38) and constant $\lambda_1$, which determines the evolution of the material parameters during cyclic loading. To understand the physical meaning of the constants $C_{p1}$ and $C_{p2}$, Eq. (36) can be integrated to find the following expression for $\varepsilon^p$ as a function of $\xi^d$:

$$\varepsilon^p = C_{p1}C_{p2} \left(1 - e^{-\frac{\xi^d}{C_{p2}}}\right) = \varepsilon^p_{\text{sat}} \left(1 - e^{-\frac{\xi^d}{C_{p2}}}\right).$$

As seen from Eq. (51) the product of $C_{p1}$ and $C_{p2}$ gives the maximum (saturation) value of the plastic strain:

$$C_{p1}C_{p2} = \varepsilon^p_{\text{sat}}.$$  \hspace{1cm} (52)

Further, recall that for stress-induced martensite $\xi^d = 2N$. Therefore, $C_{p2}$ can be estimated by enforcing the condition that after a given number of cycles the plastic strain approaches its maximum value, or more precisely, $C_{p2}$ can be found from the condition

$$e^{-\frac{2N_{\text{sat}}}{C_{p2}}} = \delta,$$  \hspace{1cm} (53)

where $N_{\text{sat}}$ is the number of cycles to reach the saturation value of plastic strain and $\delta$ is a small number. \(^4\)

Next, parameter $\lambda_1$ is considered. It can be seen from Eq. (25) that $\lambda_1$ determines the number of cycles $N_{\text{sat}}^b$ until the material parameters reach their final values. Thus, it can be found using the following equation, similar to Eq. (53):

$$e^{-2N_{\text{sat}}^b \lambda_1} = \delta.$$  \hspace{1cm} (54)

The value of the parameter $\lambda_2$ is determined similarly as the value of $\lambda_1$. For the case of stress-induced transformation, however, it is difficult to distinguish between the parameters $\lambda_1$ and $\lambda_2$. Thus in all further calculations in this work these parameters are taken to be equal.

### 3.4. Material parameters for minor loop modeling

Only one material parameter enters this last group. This is the constant $\gamma$, dictating the size of the minor hysteresis loop. A smaller value of $\gamma$ leads to smaller area occupied by the minor loop, while larger value increases the area. An experimental stress–strain curve with one minor loop is

\(^4\) Value of 0.01 has been used in the current work, however, values of up to 0.1 are also reasonable.
sufficient to estimate the value of $\gamma$. A typical value of $\gamma$ is in the range of 1–4. To illustrate the effect of this parameter on the shape of the minor loop branches, the results for different values of $\gamma$ are shown in Fig. 9.

4. Correlation with experimental data

The experimental data for NiTi undergoing cyclic loading will be simulated in this section. Two sets of experimental data will be used. First, in Section 4.1 the model will be calibrated using the results reported by Strnadel et al. (1995b) and presented in Section 2.1. Next, in Section 4.2 a set of experimental results obtained at Active Materials and Structures Laboratory at Texas A&M University will also be used to calibrate the model and the model simulations will be presented. Finally, in Section 4.3 a loading of a torque tube will also be simulated.

First the material parameters for the alloy are determined based on the experimental graphs and reported data, using the procedure described in Section 2.5. The obtained parameters are given in Table 2. It should be mentioned that there is no data on the dependence of the current maximum transformation strain $H_{\gamma_{\text{cur}}}$ on the value of the applied stress. Therefore, the data shown in Fig. 6 is used to obtain the back stress coefficients $D_i^b$. The coefficients are obtained for the value of the maximum transformation strain for high values of stress $H = 0.038$ as reported by Strnadel et al. (1995b). A polynomial of degree 5 is used in the expression for the back stress. Based on the experimental results the back stress parameters are assumed to be the same for both the initial and the final state of the material and are given by $D_1^b = 6.18 \times 10^5$ MPa, $D_2^b = -7.37 \times 10^5$ MPa, $D_3^b = 4.98 \times 10^7$ MPa, $D_4^b = -1.63 \times 10^9$ MPa and $D_5^b = 2.03 \times 10^{10}$ MPa.

4.1. Experimental data published in the literature

The experimental data for one of the NiTi alloys reported by Strnadel et al. (1995b) is used in this section. Two types of tests were performed by Strnadel et al. (1995b): the first type involved cycling, during which the material is loaded to the same value of strain during each cycle and the second type involved cycling performed to a given constant value of stress. Both types of tests have been performed at a temperature $T = 46$ °C, higher than the austenitic finish temperature $A_\gamma^f$. The experiments are performed on flat tensile specimens (strips) with cross section dimensions 1×4 mm. Prior to the testing the specimens are polished and vacuum annealed at 400 °C for 1 hr. The experimental results for NiTi alloy with composition Ti–50.9at%Ni for both loading cases are shown in the top portion of Fig. 1. Based on the dimensions of the specimen and the reported elongation the maximum value of strain during the first cycling test is estimated to be equal to $\varepsilon_{\text{max}} = 0.04$. The maximum value of the stress during the second test is reported to be $\sigma_{\text{max}} = 550$ MPa. Twenty loading–unloading cycles with a constant maximum value of strain have been performed, while the number of cycles with a constant maximum value of stress is 50.

First the material parameters for the alloy are determined based on the experimental graphs and reported data, using the procedure described in Section 2.5. The obtained parameters are given in Table 2. It should be mentioned that there is no data on the dependence of the current maximum transformation strain $H_{\gamma_{\text{cur}}}$ on the value of the applied stress. Therefore, the data shown in Fig. 6 is used to obtain the back stress coefficients $D_i^b$. The coefficients are obtained for the value of the maximum transformation strain for high values of stress $H = 0.038$ as reported by Strnadel et al. (1995b). A polynomial of degree 5 is used in the expression for the back stress. Based on the experimental results the back stress parameters are assumed to be the same for both the initial and the final state of the material and are given by $D_1^b = 6.18 \times 10^5$ MPa, $D_2^b = -7.37 \times 10^5$ MPa, $D_3^b = 4.98 \times 10^7$ MPa, $D_4^b = -1.63 \times 10^9$ MPa and $D_5^b = 2.03 \times 10^{10}$ MPa.
The initial set of material parameters has been determined using the initial transformation cycle for the case of loading up to a constant value of stress, while the final set has been determined from the stress–strain response after 50 cycles (see Fig. 1). In addition, as seen from Table 2 the parameters \( \lambda_1 \) and \( \lambda_2 \) have been selected to be the same. Their value has been determined such that the material parameters reach their saturation values after 50 cycles. As seen from Table 2, not all of the parameters change their values during the transformation cycling. Based on the experimental results the maximum transformation strain is assumed to be constant. Only the initial value of the martensitic start temperature is reported in the work by Strnad et al. (1995b), thus, here it is also assumed to remain constant. Four of the parameters—\( k^A \), \( k^M \), \( c^A \), \( c^M \)—are taken from the work of Lagoudas and Bo (1999). Since both works (Strnad et al., 1995b; Lagoudas and Bo, 1999) deal with NiTi alloys it is expected that these physical constants will be close for alloys with only slight change in the composition. As explained earlier in Section 3.2, the values of these constants have no effect on the results when the temperature during the loading–unloading cycle remains constant.

### 4.1.1. Response to cycling up to a constant value of stress

The response of the material for the cyclic loading up to a constant value of stress is presented in Fig. 10. Only the stress–strain curves for the first and 50th (last) loading cycle are shown. It can be seen that the transformation response after 50 cycles has stabilized and the plastic strain developed during the last cycle is negligible. The transformation loop has significantly evolved during the cyclic loading. First, the value of stress for the onset of phase transformation is much lower for the last loading cycle compared to the one for the first cycle. Also, the transformation

![Stress-strain response of NiTi SMA to cyclic loading up to a constant value of stress: curves for the first and 50th cycles.](image-url)
hardening during the last cycle is significantly higher than the hardening during the first cycle. The area enclosed by the transformation hysteresis loop is smaller for the last cycle than the area enclosed by the initial loop.

The plastic strain for this loading case has saturated after 50 cycles. The graph showing its evolution during the cycling is shown in Fig. 11. Further transformation cycling will result in a negligible change of the plastic strain and the shape of the transformation loop. As seen from Fig. 10, the modeling results are in a good agreement with the experimental observations. It must be pointed out that these modeling results are not predictions but simulations of the experiment, since the experimental data has been used to estimate the material parameters.

4.1.2. Response to cycling up to a constant value of strain

The response of the material during cyclic loading up to a constant value of strain is shown in Fig. 12. Two stress–strain curves are presented: the initial stress–strain curve (first cycle) as well as the stress–strain curves for the 10th cycle. It is observed from the figure that as the number of cycles increases, the value of stress at the maximum value of strain $e_{\text{max}}$ decreases. In addition, the area enclosed by the hysteresis loop also decreases. Two factors acting in concert are responsible for these effects. First, the accumulation of the residual strain contributes for the smaller hysteresis area as well as for the lower value of stress. The second factor is the evolution of the material parameters with the number of cycles. Since the value of stress for the onset of the phase transformation decreases, then the same value of strain for a later cycle will correspond to a lower value of stress than that for an earlier cycle.

The evolution of the residual plastic strain for this loading case is shown in Fig. 13. It is observed that the plastic strain has not reached a saturation value and continues to increase. This can be explained by the fact that this type of cycling results in incomplete phase transformation, and, therefore, it would take much more cycles for the plastic strain to saturate than complete transformation cycles.

In contrast to the previous case, the modeling results presented for cyclic loading up to a constant value of strain are predictions, since the experimental data for this loading case has not been used to calibrate the model. The comparison of the modeling results with the experimental curves, presented in Figs. 12 and 13 shows that the results are in a good agreement. Both the stress–strain responses as well as the plastic strain evolution are predicted with good accuracy. Also, the shape of the minor hysteresis loops predicted by
the model is very close to the shape of the experimental loops.

4.2. Experiments on large diameter NiTi actuators

In addition to the experimental results found in the literature, a set of experimental data obtained in the Active Materials and Structures Laboratory at Texas A&M University. Tensile tests were performed on large diameter (2.16 mm) Ti–49.7at%Ni wires, provided by Special Metals Corp. Prior to the testing the material was heat treated to enhance its pseudoelastic response. Different times and temperatures were considered during the heat treatment. Based on the results the material was heat treated in atmospheric air at 400°C for 5 min. Note, that the wires were received mechanically polished and no additional surface treatment was performed prior to testing.

Similarly to the previous section, two types of results were obtained. First, cyclic loading up to a constant value of stress and second, cyclic loading up to a constant value of strain, were performed. The results of the first loading case were used to calibrate the model, while the results of the second loading case were used to verify the predictions of the model.

4.2.1. Cyclic loading up to a constant value of stress

A set of experiments with cycling up to a constant stress level was performed on NiTi wires. Twenty loading/unloading cycles were performed at a temperature of 70°C above the austenitic finish temperature. The maximum stress value achieved during loading was selected to be equal to 670 MPa. A representative stress–strain result for this loading case is shown in Fig. 14. It can be seen that the evolution of the stress–strain response follows the same general trend as the stress–strain curve in the results presented in Section 4.1.1. The shape and the characteristics of the pseudoelastic loops are similar for both cases. However, it should be pointed out that the amount of plastic strain observed from the results presented in the current section is significantly lower than the one observed in Section 4.1.1. The small plastic strain, which is desirable in SMA actuator applications, is due to the different composition, cold work and heat treatment of the material.

The material parameters for the model are obtained following the procedure described in Section 2.5 and using the experimental results shown in Fig. 14. The set of parameters is summarized in Table 3. The comparison of the model simulations with the experimental stress–strain data for the first and the last cycles is shown in Fig. 15. The evolution of plastic strain with the number of cycles is also shown in Fig. 16. It can be seen that the
model captures well the characteristics of the material behavior.

4.2.2. Cyclic loading up to a constant value of strain

To test the predictive capabilities of the current model a set of experiments with cyclic loading up to a constant value of strain were performed on Ti–49.7at%Ni wires, which were from the same batch with the wires tested up to a constant value of stress (see Section 4.2.1). Twenty loading/unloading cycles were performed at a temperature of 80ºC. The maximum strain level was chosen to be \( e_{\text{max}} = 0.06 \). The stress–strain response was predicted using the model and the material parameters obtained earlier and shown in Table 3. The comparison of the model predictions and the experimental results for the first and the last loading cycles is shown in Fig. 17. It can be seen that the modeling results are in good qualitative agreement with the experimental observations.

4.3. Response of an SMA torque tube

In this section the capabilities of the model to handle loading cases beyond uniaxial loading are tested by simulating an SMA torque tube. The material parameters for NiTi SMA presented in Table 2 are used in the numerical calculations. The following dimensions of the tube are used: outer diameter \( d_o = 6.34 \text{ mm} \) and inner diameter \( d_m = \)
The reason for selecting these dimensions is to model a tube which geometrically resembles tubes available commercially. The diameters used here have also been used by Qidwai (1999) and are based on the specifications of torque tubes manufactured by Memry Corp.

Based on the small thickness of the tube wall only one quadratic element in radial direction is used. In addition, since the stress is constant in the axial direction, one element in the axial direction is sufficient to obtain accurate results. To obtain an appropriate aspect ratio, the length in the axial direction has been chosen to be 0.67 mm, equal to the wall thickness. An axisymmetric finite element with a rotational degree of freedom (element CGAX8 from the ABAQUS element library, see HKS, 1997) was used. The schematic of the mesh and the boundary conditions is shown in Fig. 18. The bottom part of the tube is fixed and rotation is applied to the top part. The maximum value of the applied rotation is $2.2 \times 10^{-2}$ rad. The rotation is applied cyclically in both direction. The loading history for one full cycle is also shown in Fig. 18. Ten full rotational loading/unloading cycles in both directions have been applied.

The stress–strain response of the tube is shown in Fig. 19 where the average shear stress in the finite element is plotted versus the average shear strain. The results obtained here are in qualitative agreement with experiment.
agreement with the results reported by Lim and McDowell (1999). Since a full set of material parameters is not reported in the original work of Lim and McDowell (1999), the results cannot be compared quantitatively. It can be seen from the results that the hysteresis loop evolves with the number of loading cycles. One significant difference, observed between these results and the uniaxial results presented in Sections 4.1.2 and 4.1.1 is the value of the plastic strain at the end of the cycling test. While the final value of the plastic strain in the uniaxial test is equal to the value of the accumulated plastic strain, the final value in the case of torsional loading is significantly smaller. This result can be explain as follows: in the case of torsional cycling loading with the loading history shown in Fig. 18(c) the direction of the loading is reversed during each cycle. Therefore, during each half of the loading cycle the direction of the plastic strain accumulation is also reversed. The result of this effect, as shown in Fig. 20 is small total plastic strain. It should be noted, however, that even in the presence of small observable plastic strain the material still changes during the cycling. As seen from Fig. 19 the hysteresis loop evolves during the cycling loading. Thus, the microstructural changes caused by the cyclic loading are taken into account by evolving the material parameters and updating the internal state variables.

5. Conclusions

The derivation of a three-dimensional constitutive model for fully dense SMAs with simultaneous evolution of transformation and plastic strains has been presented in this Part I of a two-paper series. The model accounts for development of transformation and plastic strains during stress-induced martensitic phase transformation, as well as for the evolution of shape and size of the hysteresis with repeated transformation cycling. In the current formulation of the model the following quantities are chosen as internal state variables: martensitic volume fraction, transformation strain, plastic strain, back stress and drag stress. The evolution equations for these parameters for the case of a three-dimensional loading have been presented. It should be noted that the plastic strain adopted in this work differs from the plastic strain in regular metal plasticity. The plastic strain defined in this work is induced by the martensitic phase transformation at relatively low values of the applied stress. Evolution equations for the three-dimensional plastic strain for the case of a material undergoing stress-induced martensitic phase transformation have been proposed.

In addition to the development of three-dimensional plastic strains the model is also capable of accounting for the evolution of the hysteresis loop during cyclic phase transformation. To accomplish this, evolution equations for the material parameters have been proposed based on the ideas presented by Bo and Lagoudas (1999b). For a given material, two sets of material parameters need to be established. Then, during cyclic loading, involving phase transformation, the material parameters change from the initial set to the final set according to the evolution equations for each of the parameters. Thus, after a given number of cycles, a stabilization of the material response is achieved.

Experimental results were used to estimate the material parameter and to demonstrate the pre-
dictive capabilities of the model. Two sets of experimental data were used. The first set of data is for cyclic loading of Ti–50.9at%Ni SMA alloy (Strnad et al., 1995b). The second set of experimental results is for cyclic loading of large diameter Ti–49.7at%Ni wires. The results of two types of loading were used in both cases: cyclic loading up to a constant value of stress, used to estimate the material parameters, and loading up to a constant value of strain, used to verify the capabilities of the model. In addition, to demonstrate the capabilities of the model to handle various loading cases, the loading of a torque tube has also been simulated. It has been found that due to the reversal of the loading direction during torsional loading, the observed final value of the plastic strain is significantly lower than the plastic strain observed in uniaxial experiments. The multiaxial capabilities of the model will be fully utilized in Part II (Entchev and Lagoudas, to be published) of this work, where the model will be implemented in a micromechanical averaging scheme to obtain the effective properties of porous SMAs.

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