IMPACT INDUCED PHASE TRANSFORMATION IN SHAPE MEMORY ALLOYS

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ABSTRACT

A dynamic analysis is given of the impact induced phase transformation in a shape-memory alloy rod, with a special focus on the propagation of stress waves and phase transformation fronts in the rod. The material behavior of the shape-memory alloy is modelled by a thermomechanical constitutive theory developed by Lagoudas, Bo, and Qidwai (1996), which is based on the formulation of Gibbs free energy that depends on, among other variables, the martensitic volume fraction and the transformation strain, along with evolution equations derived from a dissipation potential theory. Field equations and jump conditions of the fully coupled thermal-mechanical problem are derived to account for balances of linear momentum and energy. The equations are solved using the method of characteristic curves. The solutions are found to be associated with shocks, across which various field quantities suffer jump discontinuities. A typical solution involves two wave fronts which are initiated at the impact surface and propagate into the rod. One, travelling at the acoustic speed, separates the tranquil and disturbed regions. The other, travelling at a lower speed, separates the regions of the martensitic and austenitic phases. It is found that the stress and temperature jumps across the phase boundary can be significant. A numerical example is presented.

Keywords. A: Dynamics; Phase transformation; Shock waves; B: Polycrystalline material; C: Characteristics

1. INTRODUCTION

Due to their unique thermal and mechanical properties, shape memory alloy (SMA) materials have found numerous applications in mechanical, electronic and automotive
engineering, aerospace industry, domestic appliances, and medical devices. (See, for example, the review article by Birman (1997).) Many applications, such as SMA actuators in dynamic environments, require an understanding of the response of SMAs to dynamic loads. The problem of impact induced phase transformation in SMAs provides an excellent setting in gaining this understanding by studying the generation of phase transformations at the impact surface and the propagation of phase transformation regions in the SMA material, as well as the relations among mechanical/thermal fields, material properties and loading conditions.

A sound constitutive theory is the starting point of a successful analytical modeling, either dynamic or static. A number of constitutive models have been proposed for SMAs, and in general for materials capable of undergoing phase transformations. Some of these models are reviewed in Birman (1997).

Abeyaratne and Knowles (1993), and Abeyaratne, Kim and Knowles (1994) developed a one-dimensional thermoelastic constitutive model based on a Helmholtz free-energy function, a kinetic relation and a nucleation criterion. The free energy is associated with a multi-well potential energy function, the kinetic relation is based on thermal activation theory, and nucleation is assumed to occur at a critical value of the appropriate energy barrier. Under certain loads, the material is partitioned into domains associated with different phase wells, that are separated by sharp phase boundaries. Their constitutive theory appears to be particularly suitable to model SMA single crystals.

Various constitutive theories have been proposed based on micromechanical considerations in attempt to model polycrystalline SMAs. These models involve internal state variables which characterize the extent of phase transformation, as well as evolution equations which relate the internal variables to other state variables. Among others, Tanaka and Nagaki (1982) derived three-dimensional incremental constitutive equations that relate the time derivative of stress tensor to those of strain tensor, temperature, and the martensitic volume fraction. Tanaka, Kobayashi and Sato (1986), and Sato and Tanaka (1988) derived an evolution equation that expresses the time derivative of the martensitic volume fraction in terms of the state variables. A different form of evolution equation was proposed by Liang and Rogers (1990) and Liang (1990). Liang and Rogers (1991) also developed a three-dimensional theory involving an equivalent strain, based on
the assumption that the phase transformations are governed by the distortional energy.

To account for simultaneous phase transformation and reorientation that occur in polycrystalline SMAs under nonproportional loading or in composites with SMA components, Boyd and Lagoudas (1993, 1994, 1996) developed a three-dimensional theory in which the thermoelastic coefficients depend on the martensitic volume fraction according to the rule of mixtures, and the martensitic volume fraction depends on the effective stress calculated from the deviatoric stress tensor. This model has been applied to the analysis of metal matrix composites by Lagoudas, Bo and Qidwai (1996), and yielded solutions in a qualitative agreement with experimental data.

There have been fairly extensive analytical, experimental and numerical investigations of static and quasi-static behaviors of SMAs. On the other hand, the existing study of dynamic behaviors of SMAs is rather limited, despite a clear need for understanding. The only experimental investigation that we were able to find in the literature concerning dynamic behaviors of SMA under impact loads was given by Escobar and Clifton (1993). They carried out pressure-shear plate impact experiments on Cu-14.44Al-4.19Ni single crystals to study the kinetics of the stress-induced phase transformation. Measured velocity profiles provide several indications of the existence of a propagating phase boundary. These include a difference between the measured particle velocity and the transverse component of the projectile velocity, and the evidence that the phase boundary causes a reflected quasi-longitudinal wave and a diffracted quasi-shear wave which arrive at the rear surface of the target at expected times. By the experimental data obtained in the above work, Abeyaratne and Knowles (1997) determined, using the constitutive model they developed, the values of phase boundary velocity and driving force, as well as the kinetic law which relates these two quantities.

In a general dynamic setting, impact loading on a solid material that admits phase transformations may give rise to both conventional shock waves and traveling regions within which phase transformations are taking place. Pence (1986) gave a dynamic analysis of a semi-infinite elastic bar whose stress-strain relation is not monotonic, subjected to a monotonically increasing end load. He found that if the load is sufficiently high, a strain discontinuity emerges at the end of the bar and subsequently propagates into the interior. For long periods of time, the phase boundary velocity approaches a con-
stant. Abeyaratne and Knowles (1991) studied the dynamics of phase transformations in elastic bars with a piecewise-linear and non-monotone stress-strain relation, along with two additional constitutive requirements: a kinetic relation controlling the rate at which the phase transformation takes place and a nucleation criterion for the initiation of the phase transformation. They found that the kinetic relation and the nucleation criterion together single out a unique solution from among the infinitely many solutions that satisfy the entropy jump condition at all strain discontinuities. In a related problem, Pence (1991) studied the encounter of an acoustic shear pulse with a phase boundary in an elastic material with non-monotone stress-strain relation. In particular, he showed that a pre-existing stationary phase boundary can be set in motion by a finite amplitude shear pulse. Abeyaratne and Knowles (1994a, 1994b) extended their earlier dynamic analyses by including thermal effects. They considered the initial value problems for infinite rods in which the initial data involves two distinct phases. Both heat conduction and adiabatic cases are studied.

In the above studies, the materials are characterized by multi-well potential energy functions. Different phases of the material are associated with these wells. The regions of different phases are divided by sharp phase boundaries that are discontinuities in deformation gradient. The velocity of a phase boundary is directly related to a certain driving traction through a kinetic relation which is taken as a constitutive relation of the material. Often, certain assumptions are made to simplify the analysis.

In this paper, we study the impact problem for SMA materials using a constitutive theory developed for polycrystals. A one-dimensional problem is considered in which a semi-infinite rod is subjected to an end impact load of prescribed constant stress. One of the focuses is on the generation and propagation of phase transformation regions from the impact end and on how the structure of such transformation regions depends on the impact stress. The full thermo-mechanical coupling effect is considered, with attention being paid to both the stress and temperature distributions in the transformation regions.

In Section 2, we formulate the dynamic problem using the balance laws and the entropy inequality. Both field equations and jump conditions are derived. The constitutive theory developed by Lagoudas, Bo and Qidwai (1996) is used, which involves, for the present one-dimensional problem, 2 internal variables: the martensitic volume fraction

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and the transformation strain. The latter is directly related to the former for the particular dynamic processes considered in this work. The evolution equation for the martensitic volume fraction is presented.

Adiabatic processes are considered in Section 3. The governing equations are found to form a hyperbolic system, of which smooth solutions can be constructed by the theory of Riemann invariants and characteristic curves. The stress and temperature profiles in the phase transformation regions can be found by solving a nonlinear ordinary differential equation. For certain material behaviors and boundary conditions, the stress and temperature become discontinuous in the phase transformation regions.

A particular form of the constitutive function is considered in Section 4, for which a numerical solution of the impact problem is obtained. In particular, the velocities of the front and tail of the transition region, as well as the stress and temperature variations, are obtained as a part of the solution of the initial-boundary value problem. It is found that the stress and temperature are continuous in the transition region when the impact stress is below a certain value, and they become discontinuous above this value. A special feature of the solution is that the temperature jump can be as large as 50°K under certain conditions.

2. BASIC FORMULATION

We consider a semi-infinite rod that is subjected to an impact load at the end starting from an initial time. At the initial time, the rod occupies the positive real axis, which is taken as the reference configuration. A material particle is represented by its coordinate \( x \in \mathbb{R}_+ \) in the reference configuration. Denote by \( u(x, t) \) and \( T(x, t) \) the displacement and the absolute temperature of the material point \( x \) at time \( t \). The function \( u \) is assumed to be continuous and piecewise \( C^2 \) in \( \Omega \equiv \mathbb{R}^2_+ \), and \( T \) is assumed\(^1\) to be piecewise \( C^2 \) in

\(^1\)In the presence of heat conduction, the temperature is expected to be continuous. The effect of heat conduction, however, is small compared to that of stress power and inertia for typical impact problems. This may give rise to solutions in which the temperature changes rapidly in some narrow regions known as shocks, and changes much slower elsewhere. An idealization of such solutions can be taken by neglecting heat conduction and allowing the temperature to be discontinuous across the shocks, as is the case in adiabatic analyses.
\( \Omega \). The position of the material point \( x \) at time \( t \) is given by \( x + u(x, t) \). The strain \( \epsilon \) and the velocity \( v \) are given by

\[ \epsilon(x, t) = u_x(x, t), \quad v(x, t) = u_t(x, t), \]  

(1)

where the subscripts \( x \) and \( t \) denote the partial derivatives with respect to these variables.

In this work, we shall neglect body forces and external body heating sources. This, in the three-dimensional reality, excludes the heat conduction across the lateral surface of the rod. The local form of the balance of linear momentum gives the following equation of motion

\[ \rho v_t = \sigma_x, \]  

(2)

where \( \rho \) is the constant mass density and \( \sigma(x, t) \) the stress field which is assumed to be piecewise \( C^1 \) in \( \Omega \). The value of \( \sigma \) is a measure of the resultant axial force over the cross section of the rod. For shape memory alloys, the martensitic phase transformation is likely to be induced locally by shear type stresses. This needs to be described by a three-dimensional constitutive theory. The one-dimensional theory presented in this work is derived from a general three-dimensional theory by using the plane stress assumption, which is a reasonable approximation for slender rods with traction free lateral surfaces. The derivation is routine and omitted for brevity.

The local form of the balance of energy states

\[ \rho(e + \frac{1}{2}v^2)_t = (\sigma v - q)_x, \]  

(3)

where \( e \) is internal energy, and \( q \) the heat flux in the direction of positive \( x \).

As the second law of thermodynamics, we use the Clausius-Duhem inequality, of which the local form is

\[ \rho \eta_t + (\frac{q}{T})_x \geq 0, \]  

(4)

where \( \eta \) is entropy.

In a dynamic process, there may exist moving surfaces, across which various field quantities suffer jump discontinuities. Such a surface may be a conventional shock wave
front in mechanics, or a moving phase boundary that separates two material phases. For brevity, we shall call this surface a shock, regardless of the nature of the discontinuities. In the present one-dimensional analysis, the location of a shock can be identified by the material coordinate \( S(t) \) of the particle which the shock is crossing. The function \( S \) is assumed to be of \( C^1 \). Its time derivative, denoted by \( \dot{S}(t) \), is the reference velocity of the shock. The true velocity \( V(t) \) of the shock is given by

\[
V(t) = v(S(t), t) + [1 + \epsilon(S(t), t)]\dot{S}(t).
\]

The physical laws from which equations (2)-(4) are derived must be obeyed at a surface of discontinuity. This leads to jump conditions at shocks. We use the usual notation for the jump of a function \( f(x, t) \) at a shock \( x = S(t) \):

\[
[[f]] \equiv \lim_{\epsilon \to 0}[f(S(t) + \epsilon, t) - f(S(t) - \epsilon, t)].
\]

The balance of linear momentum, balance of energy, and Clausius-Duhem inequality at the shock \( S(t) \) imply

\[
\rho \dot{S}([[\nu]]) + [[\sigma]] = 0, \tag{5}
\]

\[
\rho \dot{S}([[e + \frac{1}{2}v^2]]) + [[\sigma v - q]] = 0, \tag{6}
\]

\[
-\rho \dot{S}([[\eta]]) + [[\frac{q}{T}]] \geq 0. \tag{7}
\]

Moreover, it follows from the continuity of the displacement that

\[
\dot{S}([[\epsilon]]) + [[\nu]] = 0. \tag{8}
\]

The rod is composed of a shape memory alloy which is capable of undergoing stress- and temperature-induced phase transformations. In this work, we shall employ a thermomechanical constitutive theory developed by Lagoudas, Bo and Qidwai (1996). This theory is for three-dimensional bodies. Here, we shall present a one-dimensional version of this theory, based on the plane stress assumption.
The constitutive theory is formulated with Gibbs free energy $G$ of the following form:

$$G = G(\sigma, T, \xi),$$

where $\xi$ is martensitic volume fraction, which is a function of $x$ and $t$, with values in $[0,1]$. The function $\xi(x,t)$ is assumed to be piecewise C$^1$ and may suffer jump discontinuity across a phase boundary that separates two material phases. The Gibbs free energy is related to the internal energy $e$ by the following equation

$$e = G + T \eta + \frac{1}{\rho} \sigma (\epsilon - \epsilon^t),$$

where $\epsilon^t$ is transformation strain. The strain $\epsilon$ and entropy $\eta$ are given by

$$\epsilon = \epsilon^t - \rho \frac{\partial G}{\partial \sigma},$$

$$\eta = -\frac{\partial G}{\partial T}.$$  

Determination of the martensitic volume fraction $\xi$ and the transformation strain $\epsilon^t$ needs additional equations. In this work, we assume that the rod is initially in austenitic phase and that each material particle undergoes only forward transformation from austenitic phase to martensitic phase. This occurs when the magnitude of the impact load is non-decreasing. In this case, the martensitic volume fraction $\xi$ and the transformation strain $\epsilon^t$ are related by

$$\epsilon^t = H \text{sgn}(\sigma) \xi,$$

where $H$ is a positive material constant corresponding to the maximum transformation strain. The evolution equation for $\xi$ is derived from a dissipation potential theory. The thermodynamic force $\pi$ conjugate to $\xi$ is defined by

$$\pi = H|\sigma| - \rho \frac{\partial G}{\partial \xi}.$$  

The forward phase transformation begins when the value of $\pi$ reaches a certain threshold.
value \( Y \). The positive material constant \( Y \) is a measure of the internal dissipation during the phase transformation. In the course of the phase transformation, the value of \( \pi \) equals \( Y \), while the value of \( \xi \) increases from 0 to 1. This transformation process can be expressed by the Kuhn-Tucker conditions

\[
(\pi - Y)\xi_t = 0, \quad \xi_t \geq 0. \tag{15}
\]

The evolution equation (15) is algebraic in nature. In the reference configuration, the rod is assumed to be stress free with constant temperature \( T_R \). Moreover, the Gibbs free energy function is assumed to be such that

\[
-\rho \frac{\partial G}{\partial \xi}(0, T_R, 0) \leq Y. \tag{16}
\]

Hence, in an initial stage of a dynamic process we have \( \pi < Y \), and the value of \( \xi \) remains zero until

\[
H|\sigma| - \rho \frac{\partial G}{\partial \xi} - Y = 0, \tag{17}
\]

signifying the start of martensitic transformation. From this point on, the value of \( \xi \) is determined by the algebraic equation (17) until the value of \( \xi \) reaches unity, signifying the completion of the transformation to martensite. Subsequently, the value of \( \xi \) remains unity. Suppose that one can solve equation (17) for \( \xi \), yielding

\[
\xi = \hat{\xi}(\sigma, T).
\]

Then the value of \( \xi \) in a dynamic process with only forward phase transformations is given by

\[
\xi(\sigma, T) = \begin{cases} 
0 & \text{if } \hat{\xi}(\sigma, T) \leq 0 \\
\hat{\xi}(\sigma, T) & \text{if } 0 < \hat{\xi}(\sigma, T) < 1 \\
1 & \text{if } \hat{\xi}(\sigma, T) \geq 1
\end{cases} \tag{18}
\]

The constitutive theory is completed by specifying the form of the heat flux \( q \). Here we employ the Fourier's law

\[
q = -kT_x, \tag{19}
\]
where \( k \) is the heat conduction coefficient.

We now rewrite the governing equations in a form convenient for mathematical treatment. By (1), (2), and (9)-(15), equation (3) can be rewritten as

\[
\rho T \eta_t - Y \xi_t = -q_x. \tag{20}
\]

By (10)-(13) and (19), we can eliminate \( u, e, \epsilon, \epsilon', \eta \) and \( q \) in (1), (2), (4)-(8) and (20), yielding

\[
H \text{sgn}(\sigma) \xi_t - \rho \left( \frac{\partial G}{\partial \sigma} \right)_t = v_x, \tag{21}
\]

\[
\rho v_t = \sigma_x, \tag{22}
\]

\[
\rho T \left( \frac{\partial G}{\partial T} \right)_t + Y \xi_t + k T_{xx} = 0, \tag{23}
\]

\[
Y T \xi_t + k T_z^2 \geq 0, \tag{24}
\]

\[
\dot{S} \left[ (H \text{sgn}(\sigma) \xi - \rho \frac{\partial G}{\partial \sigma}) \right] + [(v)] = 0, \tag{25}
\]

\[
\rho \dot{S} \left[ [v] \right] + [[\sigma]] = 0, \tag{26}
\]

\[
\rho \dot{S} \left[ [G - T \frac{\partial G}{\partial T} - \sigma \frac{\partial G}{\partial \sigma} + \frac{1}{2} v^2] \right] + [[\sigma v + k T_x]] = 0, \tag{27}
\]

\[
\rho \dot{S} \left[ \frac{\partial G}{\partial T} \right] - k \left[ \frac{T_x}{T} \right] \geq 0. \tag{28}
\]

Equations (21)-(28) are those to be analyzed in the remainder of this paper. After substituting (18) for \( \xi \) in these equations, there remain 3 field quantities to be determined: \( v, \sigma \) and \( T \). The field equations (21)-(23) must be satisfied in the region where the field quantities are smooth. At the shocks, the jump conditions (25)-(27) must be satisfied. Furthermore, inequalities (24) and (28) must be satisfied in order for a solution to be physically acceptable. We note that in virtue of (15), inequality (24) holds if \( k \) is non-negative.

The governing equations may be simplified for special dynamic processes. For instance, if the rod is very thin and placed in a thermal bath of constant temperature, isothermal processes become a good approximation. In this case, the energy balance equation is satisfied by introducing a suitable external body heating (along the length of
the rod) term, and the temperature $T$ ceases to be an unknown variable.

On the other hand, in an impact problem, the heat conduction is likely to be insignificant because of rapid loading. It is then a reasonable approximation to assume that the processes are adiabatic with the heat conduction terms being ignored. We now turn our attention to such a case.

3. ADIABATIC PROCESSES

The system of partial differential equations (21)-(23) is mixed hyperbolic-parabolic, due to the presence of the heat conduction term $T_{xx}$. For an adiabatic process, this term vanishes. Under certain conditions, this leads to a hyperbolic system, of which the solution can be constructed by a well developed theory.

To this end, we define

$$
\varepsilon(\sigma, T) \equiv H \text{sgn}(\sigma) \xi(\sigma, T) - \rho \frac{\partial G}{\partial \sigma}(\sigma, T, \xi(\sigma, T)),
$$

$$
\eta(\sigma, T) \equiv -\frac{\partial G}{\partial T}(\sigma, T, \xi(\sigma, T)).
$$

Here we have used the same notation for different functions whose values are the same physical quantities. Dropping the heat conduction terms in (23), (27) and (28), we find that equations (21)-(23) and (25)-(28) become

$$
\varepsilon_\sigma \sigma_t + \varepsilon_t T_t = v_z,
$$

$$
\rho v_t = \sigma_z,
$$

$$
(\rho T \eta_\sigma - Y \xi_\sigma) \sigma_t + (\rho T \eta_T - Y \xi_T) T_t = 0,
$$

$$
\mathcal{S}[[\varepsilon]] + [[v]] = 0,
$$

$$
\rho \mathcal{S}[[v]] + [[\sigma]] = 0,
$$

$$
\rho \mathcal{S}[[G + T \eta + \frac{1}{\rho} \sigma(\varepsilon - H \text{sgn}(\sigma) \xi) + \frac{1}{2} v^2]] + [[\sigma v]] = 0,
$$

$$
-\mathcal{S}[[\eta]] \geq 0,
$$

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where functions $\xi(\sigma, T), \varepsilon(\sigma, T)$ and $\eta(\sigma, T)$ are defined in (18), (29) and (30), and function $G$ is evaluated at $(\sigma, T, \xi(\sigma, T))$. Inequality (24) is satisfied automatically for an adiabatic process.

The system of first order partial differential equations (31)-(33) is hyperbolic if

$$
\varepsilon_\sigma - \frac{\varepsilon_T (\rho T \eta_\sigma - Y \xi_\sigma)}{\rho T \eta_T - Y \xi_T} > 0.
$$

(37)

The solution of a hyperbolic system is well understood. A smooth solution can be constructed by the theory of Riemann invariants and characteristic curves. A comprehensive account of the theory can be found in Courant and Friedrichs (1956). Under the condition (37), the coefficient matrix of the system has 3 distinct real eigenvalue/eigenvector pairs. From them, 3 families of characteristic lines can be constructed, along which certain combinations of the unknown variables are invariant. Indeed, defining

$$
B(\sigma, T) \equiv \frac{\rho T \eta_\sigma - Y \xi_\sigma}{\rho T \eta_T - Y \xi_T},
$$

(38)

$$
C(\sigma, T) \equiv [\rho(\varepsilon_\sigma - \varepsilon_T B)]^{-\frac{1}{2}},
$$

(39)

we can readily verify that

$$
B \frac{d\sigma}{dt} + \frac{dT}{dt} = 0 \quad \text{along the curve} \quad \frac{dx}{dt} = 0,
$$

(40)

$$
\rho C \frac{dv}{dt} - \frac{d\sigma}{dt} = 0 \quad \text{along the curve} \quad \frac{dx}{dt} = C,
$$

(41)

$$
\rho C \frac{dv}{dt} + \frac{d\sigma}{dt} = 0 \quad \text{along the curve} \quad \frac{dx}{dt} = -C.
$$

(42)

Solutions can be obtained by integrating equations (40)-(42) along the characteristic lines when suitable initial and boundary conditions are given. The initial conditions for the rod considered in this paper are

$$
v(x, 0) = 0, \quad \sigma(x, 0) = 0, \quad T(x, 0) = T_R, \quad \xi(x, 0) = 0, \quad 0 \leq x < \infty.
$$

(43)

We assume that from the initial time the rod is subjected to an impact load of prescribed
constant stress $\sigma_0$ at the end, and that the impact end is thermally insulated. The boundary conditions are then

$$
\sigma(0, t) = \sigma_0, \quad T_z(0, t) = 0, \quad 0 < t < \infty.
$$

(44)

We remark that the following analysis can be made appropriate for other types of boundary conditions, for example, one that prescribes the impact velocity.

For the given constant initial and boundary conditions (43) and (44), a solution must be constant in the domains of dependence of the $x-$ and $t-$axes. The characteristic lines in these domains are parallel straight lines. Furthermore, in the adjacent domains, the solution represents simple waves for which the characteristic lines $dx/dt = C$, referred to as $C$ curves, are straight lines. Since the origin is the only point on $\partial \Omega$ where the solution changes its values, all $C$ curves are rays centered at the origin (centered fans).

Now we investigate the solutions corresponding to centered fans. These are the solutions of the form

$$
v(x, t) = \hat{v}(\frac{x}{t}), \quad \sigma(x, t) = \hat{\sigma}(\frac{x}{t}), \quad T(x, t) = \hat{T}(\frac{x}{t}).
$$

(45)

Substitution of (45) into (31)-(33) yields

$$
\begin{pmatrix}
1 & \epsilon_\sigma x/t & \epsilon_T x/t \\
px/t & 1 & 0 \\
0 & B & 1
\end{pmatrix}
\begin{pmatrix}
\hat{v}' \\
\hat{\sigma}' \\
\hat{T}'
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
$$

(46)

A nontrivial solution exits only if the determinant of the coefficient matrix in (46) vanishes, that is, only if

$$
C(\sigma, T) = \frac{x}{t}.
$$

(47)

Moreover, the third equation of (46) and the initial conditions (43) lead to the following initial value problem of a single equation

$$
\frac{dT}{d\sigma} = -B(\sigma, T), \quad T|_{\sigma=0} = T_R.
$$

(48)
The theory concerning the existence and uniqueness of the solution of this initial value problem is well known. See, for example, Ince (1956). We denote by \( T = \hat{T}(\sigma) \) the solution of (48) and define
\[
\hat{C}(\sigma) \equiv C(\sigma, \hat{T}(\sigma)). \tag{49}
\]
From (47) and (49), we observe that \( \hat{C}(\sigma) \) is the velocity of the locus of constant stress \( \sigma \). In particular, it gives the velocity of a wave front of stress level \( \sigma \).

This velocity is directly related to the slope of the stress-strain curve. Indeed, if we define
\[
\hat{\varepsilon}(\sigma) \equiv \varepsilon(\sigma, \hat{T}(\sigma)), \tag{50}
\]
\( \varepsilon(\sigma, T) \) being given by (29), then we have
\[
\begin{align*}
\hat{\varepsilon}'(\sigma) & = \varepsilon_\sigma + \varepsilon_T \hat{T}'(\sigma) \\
& = \varepsilon_\sigma - \varepsilon_T B \\
& = \rho^{-1} [\hat{C}(\sigma)]^{-2}
\end{align*}
\]
Here use has been made of (48)\(_1\) and (39). We hence arrive at
\[
\hat{C}(\sigma) = [\rho \hat{\varepsilon}'(\sigma)]^{-1/2}, \tag{51}
\]
which is a generalization of the well know result that the wave speed equals the square root of the stiffness divided by the density.

In a pure mechanical setting, the value of \( \hat{C}(\sigma) \) is proportional to the square root of the elastic stiffness. The material is said to have increasing stiffness if \( \hat{C}(\sigma) \) is monotone increasing in \( |\sigma| \), and is said to have decreasing stiffness if \( \hat{C}(\sigma) \) is monotone decreasing in \( |\sigma| \). We shall continue using these terminologies, despite that in the present work the mechanical effect is fully coupled with the thermal effect, as well as the effect of material phase transformation.

If the function \( \hat{C}(\sigma) \) is strictly monotone decreasing in \( |\sigma| \), one can construct families of \( C \) lines that cover the entire domain \( \Omega \) without intersecting each other, as depicted in Fig. 1. The domain \( \Omega \) is divided into 3 regions. In Region I, that corresponds to the
tranquil region, the $C$ lines are parallel straight lines. Regions II and III, corresponding to the disturbed regions, are formed by a centered fan and a family of parallel straight lines, respectively. The complete solution of the impact problem is given by

$$v(x,t) = \begin{cases} v_0 & \text{if } x \leq \tilde{C}(\sigma_0)t \\ \hat{v}(x/t) & \text{if } \tilde{C}(\sigma_0)t < x < \tilde{C}(0)t \\ 0 & \text{if } x \geq \tilde{C}(0)t \end{cases}$$

(52)

$$\sigma(x,t) = \begin{cases} \sigma_0 & \text{if } x \leq \tilde{C}(\sigma_0)t \\ \hat{\sigma}(x/t) & \text{if } \tilde{C}(\sigma_0)t < x < \tilde{C}(0)t \\ 0 & \text{if } x \geq \tilde{C}(0)t \end{cases}$$

(53)

$$T(x,t) = \begin{cases} T_0 & \text{if } x \leq \tilde{C}(\sigma_0)t \\ \hat{T}(x/t) & \text{if } \tilde{C}(\sigma_0)t < x < \tilde{C}(0)t \\ T_R & \text{if } x \geq \tilde{C}(0)t \end{cases}$$

(54)

Fig. 1. Characteristic lines for a rod of decreasing stiffness.
where

\[ \dot{\sigma}(\frac{x}{t}) = \tilde{C}^{-1}(\frac{x}{t}), \] (55)

\[ \dot{T}(\frac{x}{t}) = \tilde{T}(\dot{\sigma}(\frac{x}{t})), \] (56)

\[ \dot{v}(\frac{x}{t}) = \int_{x/t}^{\tilde{C}(0)} \frac{1}{\rho y} \dot{\sigma}'(y) dy, \] (57)

\[ v_0 = \dot{v}(\tilde{C}(\sigma_0)), \quad T_0 = \tilde{T}(\tilde{C}(\sigma_0)). \]

The solution (52)-(54) is smooth in the sense that the velocity, strain, stress and temperature are continuous in \( \Omega \). Upon the impact, a continuous transition region propagates into the rod. The front of the transition region moves at speed \( \tilde{C}(0) \), and the tail at speed \( \tilde{C}(\sigma_0) \). Ahead of the transition region, the rod is undisturbed. Behind the transition region are constant fields, with the stress being the prescribed impact stress \( \sigma_0 \).

A limiting case is linear thermoelasticity for which \( \tilde{C} \) is constant. In this degenerate case, the two straight lines \( x = \tilde{C}(\sigma_0)t \) and \( x = \tilde{C}(0)t \) coincide. This forces the transition region to disappear, rendering a shock front across which the velocity, stress and temperature suffer jump discontinuities. Such a piecewise constant solution can be readily found from jump conditions (34)-(36).

The construction of characteristic lines shown in Fig. 1 breaks down if the rod fails to have decreasing stiffness. For some shape memory alloy materials, \( \tilde{C}(\sigma) \) is approximately constant before the onset of the stress-induced phase transformation, decreasing in \( |\sigma| \) in an initial stage of the transformation, and then increasing later during or after the completion of the transformation. Such a material will be said to have decreasing-increasing stiffness, and we shall consider this material behavior in the remainder of this section. When \( \tilde{C}(\sigma) \) becomes increasing in \( |\sigma| \), the construction of a smooth solution is no longer possible as the characteristic lines in Region II intersect those in Region III. At the intersection, the velocity, stress and temperature may be discontinuous. A shock is then generated.

We now construct a solution for the material of decreasing-increasing stiffness described above. We assume that the impact stress \( \sigma_0 \) is sufficiently large so that \( \tilde{C}(\sigma) \)
becomes strictly increasing before $\sigma$ reaches $\sigma_0$. The characteristic lines of this solution are depicted in Fig. 2. Region I is again the tranquil region, and Regions II and III are disturbed. The boundary $x = Ct$ between Regions II and III is the shock line. The characteristic lines in Region III intersect the shock line, while the characteristic lines in Region II approach the shock line. The corresponding solution is of the form

$$v(x,t) = \begin{cases} v_0 & \text{if } x < Ct \\ \dot{\theta}(x/t) & \text{if } Ct < x < \hat{C}(0)t \\ 0 & \text{if } x \geq \hat{C}(0)t \end{cases}$$  \quad (58)

$$\sigma(x,t) = \begin{cases} \sigma_0 & \text{if } x < Ct \\ \dot{\theta}(x/t) & \text{if } Ct < x < \hat{C}(0)t \\ 0 & \text{if } x \geq \hat{C}(0)t \end{cases}$$  \quad (59)

$$T(x,t) = \begin{cases} T_0 & \text{if } x < Ct \\ \dot{T}(x/t) & \text{if } Ct < x < \hat{C}(0)t \\ T_R & \text{if } x \geq \hat{C}(0)t \end{cases}$$  \quad (60)

Fig. 2. Characteristic lines for a rod of decreasing-increasing stiffness.
where functions \( \hat{\sigma}, \hat{T} \) and \( \hat{v} \) are again given by (55)-(57), and the constants \( C, v_0 \) and \( T_0 \) are to be determined by the jump conditions (34)-(36) as follows.

We define
\[
\begin{align*}
    v_1 & \equiv \hat{v}(C), \quad \sigma_1 \equiv \hat{\sigma}(C), \quad T_1 \equiv \hat{T}(C), \quad \xi_a \equiv \xi(\sigma_a, T_a), \\
    \epsilon_a & \equiv \epsilon(\sigma_a, T_a), \quad \eta_a \equiv \eta(\sigma_a, T_a), \quad G_a \equiv G(\sigma_a, T_a, \xi_a), \quad \alpha = 0, 1.
\end{align*}
\]

The jump conditions at the shock \( x = Ct \) can be written as
\[
\begin{align*}
    C(\epsilon_1 - \epsilon_0) + v_1 - v_0 &= 0, \tag{63} \\
    \rho C(v_1 - v_0) + \sigma_1 - \sigma_0 &= 0, \tag{64} \\
    C \{ \rho(G_1 - G_0 + T_1 \eta_1 - T_0 \eta_0) + \sigma_1 [\epsilon_1 - H \text{sgn}(\sigma_1) \xi_1] - \sigma_0 [\epsilon_0 - H \text{sgn}(\sigma_0) \xi_0] + \frac{\rho}{2} (v_1^2 - v_0^2) \} + \sigma_1 v_1 - \sigma_0 v_0 &= 0, \tag{65}
\end{align*}
\]

We can eliminate \( v_1 \) and \( v_0 \) in (63)-(65), yielding
\[
\rho C^2(\epsilon_1 - \epsilon_0) = \sigma_1 - \sigma_0, \tag{66}
\]
\[
\rho(G_1 - G_0 + T_1 \eta_1 - T_0 \eta_0) - H(|\sigma_1|\xi_1 - |\sigma_0|\xi_0) + \frac{1}{2} (\epsilon_1 + \epsilon_0)(\sigma_1 - \sigma_0) = 0. \tag{67}
\]

Equation (66) states the well-known result that the reference velocity of a shock equals the square root of the ratio of the stress jump and the strain jump divided by the density.

By (51), (55) and (61), equation (66) can be written in an equivalent form
\[
\epsilon_1 + (\sigma_0 - \sigma_1) \hat{\epsilon}(\sigma_1) = \epsilon_0. \tag{68}
\]

This equation states that the straight line connecting the stresses and strains on the two sides of the shock is tangent to the static stress-strain curve at \( \sigma = \sigma_1 \), as shown in Fig. 3. Moreover, by (17), (29), (38), (48), (50) and (61), equation (67) can be rewritten as
\[
\frac{1}{2} (\epsilon_0 + \epsilon_1)(\sigma_0 - \sigma_1) - \int_{\sigma_1}^{\sigma_0} \hat{\epsilon}(\sigma) d\sigma = [\rho G(\sigma_0, T, \xi(\sigma_0, T)) + \rho T \eta(\sigma_0, T) - H|\sigma_0|\xi(\sigma_0, T)]_{T=\tilde{T}_0}^{T=T_0}.
\]

The left-hand side of (69) is the difference between the area of the trapezoid determined
by \((\epsilon_1, \sigma_1), (\epsilon_0, \sigma_0), (0, \sigma_0)\) and \((0, \sigma_1)\) and the area on the left of the static stress-strain curve from \(\sigma_1\) to \(\sigma_0\). This is also the difference between the "+" area and the "−" area, as marked in Fig. 3. The right-hand side of (69) would be zero and the jump condition (69) would be violated if \(T_0 = \tilde{T}(\sigma_0)\), i.e., if \((\epsilon_0, \sigma_0)\) were on the static stress-strain curve. It then follows that for a rod of decreasing-increasing stiffness, the stress-strain relation in an impact experiment is in general different from that in a static experiment where all fields are homogeneous.

![Stress-strain curve](image)

Fig. 3. Stress-strain curve and a geometric interpretation of the energy balance at a shock solution.

Abeyaratne and Knowles (1991, 1993, 1994a) have interpreted the left-hand side of (69) as a driving traction acting on the phase boundary. The shocks they considered, however, are isothermal. As a result, the points \((\epsilon_0, \sigma_0)\) and \((\epsilon_1, \sigma_1)\) are on the stress-strain curve. They also introduced a kinetic relation that directly connects the driving traction to the propagation velocity of the phase boundary.

We now examine uniqueness of shock solutions. There are two unknowns \(T_0\) and \(\sigma_1\) in the nonlinear algebraic equations (68) and (69). We observe that the left-hand sides of (68) and (69) depend only on \(\sigma_1\), and the right-hand sides only on \(T_0\). It follows from
an elementary argument that the solution of these equations is unique if the determinant of the matrix formed by the derivatives the two sides does not change sign in the $T_0 - \sigma_1$ plane, that is, if

\[(\sigma_0 - \sigma_1)\dot{\varepsilon}''(\sigma_1)[Y\dot{\xi}_T(\sigma_0, T_0) - \rho T_0 \eta_T(\sigma_0, T_0) + \frac{1}{2}(\sigma_1 - \sigma_0)\epsilon_T(\sigma_0, T_0)] \neq 0.\] (70)

For some shape memory alloy materials, the function $\dot{C}(\sigma)$ is constant before the onset of the phase transformation. For the solution of such materials, the characteristic lines are the same as those shown in Fig. 2. However, the characteristic line $x = \dot{C}(0)t$ now represents a shock line across which the fields are discontinuous, and a material point begins to undergo the phase transformation immediately at the arrival of the wave front due to the impact. The solution (58)-(60) needs to be modified accordingly by using the jump conditions (34)-(36) at $x = \dot{C}(0)t$.

Another remark here is that the martensitic phase transformation is not necessarily completed in Region II, that is, the value of $\xi_1$ may be less than 1. This is because the stiffness may become increasing in strain before the completion of the phase transformation. Even if the stiffness is monotone decreasing in the entire course of the transformation, the jump may occur before reaching the full martensite if the impact stress is sufficiently large. In fact, when the stress-strain relation is linear before the onset of the phase transformation it is possible that the jump occurs at the beginning of the transformation, that is, $\xi_1 = 0$. In this case, the fields are constant in Region II as well, rendering the entire solution piecewise constant.

In the next section, we examine the solution for a specific form of the constitutive function.

4. A SMA CONSTITUTIVE FUNCTION

We consider a specific form of the constitutive function of polycrystalline shape memory alloys, which is a special case, with a slight modification, of the constitutive model
derived by Lagoudas, Bo and Qidwai (1996). The Gibbs free energy is of the form

\[ G(\sigma, T, \xi) = -\frac{1}{2\rho}[S^A + \xi(S^M - S^A)]\sigma^2 - \frac{\alpha}{\rho}\sigma(T - T_R) - \frac{c}{2T_R}(T - T_R)^2 \]

\[ -[\eta_R^A + \xi(\eta_R^M - \eta_R^A)]T + e_R^A + \xi(e_R^M - e_R^A) + \frac{1}{\rho}f(\xi), \quad (71) \]

where \(\alpha\) is the thermal expansion coefficient, \(c\) the specific heat, \(S^A\) and \(S^M\) the elastic compliances in the austenitic phase and martensitic phase, respectively, \(\eta_R^A\) and \(\eta_R^M\) the reference specific entropies, \(e_R^A\) and \(e_R^M\) the reference specific internal energies, and \(f(\xi)\) the hardening function which physically represents the free energy of mixing. Here, it has been assumed that the thermal expansion coefficient and specific heat are constant during the phase transformation. In Lagoudas, Bo and Qidwai (1996), the term of specific heat involves a logarithmic function, as is the case for an ideal gas. Here we have taken an appropriate Taylor expansion for this term to make the stress-strain relation completely linear in the martensitic and austenitic phases.

For simplicity, a polynomial representation of the hardening function is used. For the forward phase transformations considered in this paper, we have

\[ f(\xi) = \frac{1}{2}\rho b^M \xi^2 + [\mu + \frac{1}{4}(\rho b^A - \rho b^M)]\xi, \quad (72) \]

where \(\mu\), \(b^A\) and \(b^M\) are material constants. By (71) and (72), inequality (16) states

\[ \rho \Delta \eta_R T_R - \rho \Delta e_R - \mu - \frac{1}{4}(\rho b^A - \rho b^M) \leq Y, \quad (73) \]

where

\[ \Delta \eta_R \equiv \eta_R^M - \eta_R^A, \quad \Delta e_R \equiv e_R^M - e_R^A. \]

Inequality (73) gives the restriction on the reference temperature \(T_R\) under which the rod remains in the austenitic phase in the reference configuration. Substituting (71) and (72) into (17), we find that function \(\hat{\xi}(\sigma, T)\) appearing in (18) is given by

\[ \hat{\xi}(\sigma, T) = \frac{1}{\rho b^M}[\frac{1}{2}\Delta S\sigma^2 + H|\sigma| + \rho \Delta \eta_R T_R - \rho \Delta e_R - \mu - \frac{1}{4}(\rho b^A - \rho b^M) - Y], \quad (74) \]
where
\[ \Delta S \equiv S^M - S^A. \]

For numerical solutions, we choose the reference temperature to be \( T_R = 315^\circ K \), and choose the following values of the material constants for typical equiatomic NiTi SMA from Lagoudas, Bo and Qidwai (1996):

\[ E^A = 70 \times 10^3 \text{MPa}, \quad E^M = 30 \times 10^3 \text{MPa}, \quad \alpha = 10^{-5}/^\circ K, \quad H = 0.05, \]

\[ \rho = 6.450 \text{kg/m}^3, \quad \rho \Delta \eta_R = -0.35 \text{MPa/}^\circ K, \quad \rho b^A = 7.0 \text{MPa}, \quad \rho b^M = 5.25 \text{MPa}, \]

\[ \mu + \rho \Delta \epsilon_R = -106 \text{MPa}, \quad Y = 3.76 \text{MPa}, \quad \rho c = 2.12 \text{MPa/}^\circ K. \]

In this section, the prescribed impact stress is taken to be compressive, that is, \( \sigma_0 < 0 \). Hence, the stress and strain in the rod are non-positive. Substituting (71) into (29) and (30), we find that

\[ \epsilon(\sigma, T) = S^A \sigma + \alpha (T - T_R) + (\Delta S \sigma - H) \xi(\sigma, T), \quad (75) \]

\[ \eta(\sigma, T) = \frac{\sigma}{\rho} + \frac{c}{T_R} (T - T_R) + \eta_R^A + \Delta \eta_R \xi(\sigma, T), \quad (76) \]

where
\[ \Delta \eta_R \equiv \eta_R^M - \eta_R^A. \]

The static stress-strain curve, obtained from (50) and (75) with \( \tilde{T}(\sigma) \) being determined from the complete solution is shown as the solid line in Fig. 4. The two stress levels \( \sigma_A = 165 \text{ MPa} \) and \( \sigma_M = 586 \text{ MPa} \) correspond to the start and finish of the martensitic phase transformation. The stress-strain relation is linear in the austenitic and martensitic phases (\( |\sigma| \leq \sigma_A \) and \( |\sigma| \geq \sigma_M \)), but is nonlinear in the phase transformation region (\( \sigma_A < |\sigma| < \sigma_M \)) since the solution of the differential equation (48) is not a linear...
function.

![Graph showing stress-strain relation](image)

Fig. 4. Stress-strain relation.

Also included in Fig. 4. are the isothermal stress-strain curves as dashed lines for three temperature values. It is observed that the hardening during the adiabatic phase transformation is much stronger than that in isothermal phase transformation. This contributes, among other things, to a relatively high propagation speed of the transformation region, as discussed below.

For the material behavior described in Fig. 4, the stiffness is monotone decreasing if the maximum absolute value of the stress does not exceed \( \sigma_M \), and is decreasing/increasing otherwise. The solution for the impact problem can be constructed by the method discussed in the previous section. The characteristic lines are given by Fig. 1 if \( |\sigma_0| < \sigma_M \), and by Fig. 2 if \( |\sigma_0| > \sigma_M \). However, the characteristic line \( x = \tilde{C}(0)t \) is now a shock line since the stress-strain-entropy-temperature relation is linear in the region \( |\sigma| < \sigma_A \). Moreover, the stress and the temperature are constant in a part of the centered fan II, that is adjacent to I. This corresponds to the corner point \( |\sigma| = \sigma_A \).
in Fig. 4, across which the slope of the stress-strain curve, and therefore the slope of the characteristic lines, has a sudden change with the stress and the temperature being constant. When $|\sigma_0| > \sigma_M$, the complete solution is of the form

\[
\sigma(x,t) = \begin{cases} 
\sigma_0 & \text{if } x < C_1 t \\
\dot{\sigma}(x/t) & \text{if } C_1 t < x \leq \dot{C}(\sigma_A)t \\
\sigma_A & \text{if } \dot{C}(\sigma_A)t < x < C_2 t \\
0 & \text{if } x > C_2 t
\end{cases}
\]  
(77)

\[
T(x,t) = \begin{cases} 
T_0 & \text{if } x < C_1 t \\
\dot{T}(x/t) & \text{if } C_1 t < x \leq \dot{C}(\sigma_A)t \\
T_2 & \text{if } \dot{C}(\sigma_A)t < x < C_2 t \\
T_R & \text{if } x > C_2 t
\end{cases}
\]  
(78)

Here we have relabeled the characteristic lines of importance, as shown in Fig. 5.

![Characteristic lines](image)

Fig. 5. Characteristic lines.

The functions $\dot{\sigma}$ and $\dot{T}$, as well as the constants $\sigma_A, T_0, T_2, C_1$ and $C_2$, are determined with the procedure described below.
The characteristic line \( x = C_2 t \) separates the tranquil and disturbed regions. On either side of the line, the material is in the austenitic phase. Hence, \( \sigma_A, T_2 \) and \( C_2 \) are determined by the equation \( \dot{\xi}(\sigma_A, T_2) = 0 \) and the jump conditions (66) and (67), evaluated at \( (\sigma, T) = (0, T_R) \) and \( (\sigma, T) = (\sigma_A, T_2) \). That is

\[
\rho C_2^2 \varepsilon(\sigma_A, T_2) = \sigma_A, \tag{79}
\]

\[
\rho [G(\sigma_A, T_2, 0) - G(0, T_R, 0) + T_2 \eta(\sigma_A, T_2) - T_R \eta(0, T_R)] + \frac{1}{2} \sigma_A \varepsilon(\sigma_A, T_2) = 0, \tag{80}
\]

where \( G(\sigma, T, \xi), \varepsilon(\sigma, T) \) and \( \eta(\sigma, T) \) are given by (71), (75) and (76). The uniqueness of the solution of the algebraic equations (79) and (80) has been discussed in the preceding section. It is found that for the particular constitutive function considered in this section, the left-hand side of (70) is always negative. Hence, equations (79) and (80) have a unique solution. In fact, they can be solved to give the explicit expressions of \( \sigma_A, T_2 \) and \( C_2 \).

Next, the solution \( \dot{\sigma}(x/t) \) and \( \dot{T}(x/t) \) in the region of phase transformation can be found by solving (47) and (48), along with the initial condition

\[
T|_{\sigma=\sigma_A} = T_2,
\]

where \( C(\sigma, T) \) and \( B(\sigma, T) \) are given by (38), (39), and (74)-(76). The solution of this initial value problem of the first order non-linear ordinary differential equation is found numerically. We also find

\[
\ddot{C}(\sigma_A) = C(\sigma_A, T_2).
\]

For the values of the material constants chosen above, \( \ddot{C}(\sigma) \) defined in (49) is found to be strictly decreasing in \(|\sigma|\) during the entire course of phase transformation, that is, the stiffness is decreasing up to the completion of the phase transformation. The solution of the initial value problem terminates at a point \( \sigma = \sigma_1 \), which is determined in the following way: If \( \dot{\xi}(\sigma_0, \dot{T}(\sigma_0)) \leq 1 \), then \( \sigma_1 = \sigma_0 \). In this case, the solution is continuous at \( x = C_1 t \) and the martensitic phase transformation is either incomplete (when \( \dot{\xi}(\sigma_0, \dot{T}(\sigma_0)) < 1 \)) or just completed (when \( \dot{\xi}(\sigma_0, \dot{T}(\sigma_0)) = 1 \)) at the impact end. On the other hand, if \( \dot{\xi}(\sigma_0, \dot{T}(\sigma_0)) > 1 \), the solution is discontinuous at \( x = C_1 t \), and \( \sigma_1 \)
is determined by the jump conditions (66) and (67) in connection with (62). In either case, we have

\[ C_1 = \dot{C}(\sigma_1) = C(\sigma_1, \dot{T}(\sigma_1)). \]

The calculated stress and temperature profiles are plotted in Fig. 6 for various impact stress levels.

When the impact stress is small, there is only one shock front, travelling at the acoustic speed 3.30 km/s, that separates the tranquil and disturbed regions. The stress and the temperature are constant in each region and suffer a jump discontinuity at the shock front, as shown in Fig. 6 (a) and (b). The temperature jump is rather small, approximately 0.15°K when \( \sigma_0 = -100 \) MPa. The entire rod is in austenitic phase. This solution is identical to that in linear thermoelasticity.

When the impact stress \( |\sigma_0| \) exceeds \( \sigma_A \) (approximately 165 MPa), a transition region starts forming, as shown in Fig. 6 (c) and (d), in which a partial martensitic phase transformation is in progress. The stress and the temperature are still discontinuous at the acoustic front, but are continuous in the transition region, which forms a second moving front travelling at a lower speed (1.10 km/s). The stress and the temperature are piecewise constant except in the transition region.

As the value of \( |\sigma_0| \) increases, the extent of the martensitic transformation increases. At the same time, the variations of stress and temperature across the transition region increase, while the jumps at the acoustic front remain constant (165 MPa for the stress, and 0.25°K for the temperature), as shown in Fig. 6 (e) and (f). The speed of the head of the transition region also remains constant. However, the speed of the tail decreases.

The speed of the tail of the transition region reaches a minimum value 0.911 km/s at the impact stress \( |\sigma_0| = 586 \) MPa (which is \( \sigma_M \) in Fig. 4). At this point, the martensitic phase transformation is just completed behind the transition region. When \( |\sigma_0| \) increases further, the stress and the temperature become discontinuous at the tail of the transition region, forming a second shock, as shown in Fig. 6 (g) and (h). The speed of this shock increases as \( |\sigma_0| \) increases, while the speed of the head of the transition region still remains constant.
Fig. 6. Stress and temperature profiles at various impact stress levels.
When the impact stress $|\sigma_0|$ reaches 713 MPa, the second shock catches up with the head of the transition region. The transition region is thus absorbed by the shock. From this point on, the stress and the temperature are piecewise constant, as shown in Fig. 6 (i) and (j). In this final case, as well as in the first case (a) and (b), the solution can be found by solving a system of algebraic equations derived from jump conditions. We also note that the temperature jump across the second shock can be rather large, for example, 56.5°K when $\sigma_0 = -800$ MPa.

Of theoretical and practical interest is the velocity of the transition region, as well as its dependence on the impact stress. The results presented above can be visualized from the stress-strain curve sketched in Fig. 7. No martensitic phase transformation occurs when $|\sigma_0| < \sigma_A$. When $\sigma_A < |\sigma_0| < \sigma_M$, a continuous transition region, within which the phase transformation is taking place, is initiated at the impact end and propagates into the rod. The head of the transition region corresponds to point A in Fig. 7. By equation (51), the speed of the head is determined by the slope of the stress-strain curve (phase transformation portion, i.e. segment $AM$) at point A. The speed of the tail of the transition region, on the other hand, is determined by the slope at the point of the impact stress, for example, point B when $|\sigma_0| = 200$ MPa, and point C when $|\sigma_0| = 400$ MPa. Since the stiffness is decreasing slightly in the phase transformation region for the given data, the speed of the tail decreases slightly as the impact stress increases. The minimum tail speed is attained at point M, corresponding to $|\sigma_0| = \sigma_M$. When $|\sigma_0|$ exceeds $\sigma_M$, the solution no longer reaches point M. Instead, it jumps from some point before M to the point of the impact stress, for example, from point D to point E when $|\sigma_0| = 600$ MPa. A second shock is thus formed. By (66), the propagating speed of the shock is determined by the slope of straight line connecting these two points. The exact location of the jumping point is determined by the jump conditions. As $|\sigma_0|$ increases, the jumping point moves backwards toward point A, the slope of the connecting line increases, and the speed of the shock increases. Eventually, the connecting line becomes tangent to the stress-strain curve at point A. From this point on, the jump is from point A to the point of the impact stress, such as point F when $|\sigma_0| = 800$ MPa. The speed of the shock continues to increase in $|\sigma_0|$, and approaches an asymptotic value determined
by the stiffness of the martensitic phase.

Fig. 7. Illustration of the propagation velocities of the transition region as related to the stress-strain curve.

As a concluding remark, we would like to comment on the quantitative results obtained in this section. The calculated velocity of the transition region (about 1 km/s) is considerably higher than those reported by Escobar and Clifton (1993) (under 100m/s). However, we feel that a direct comparison may not be meaningful, as a single crystal SMA was used in the experiments while the constitutive model used in the present work is for polycrystalline SMAs. The seemingly high phase velocity is associated with significant hardening of the material during the phase transformation, as illustrated in Fig. 4. Also, pressure-shear plate impact was used in the experiment of Escobar and Clifton, while axial normal impact is considered in the present work. Nevertheless, the main purpose of the present analysis has been to demonstrate the solution method and to capture the main features of the solution. A work aiming at correlating model predictions with experimental results is in progress and will be reported elsewhere.
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