

**A THERMODYNAMICAL CONSTITUTIVE MODEL FOR
SHAPE MEMORY MATERIALS.
PART II. THE SMA COMPOSITE MATERIAL**

James G. Boyd* and Dimitris C. Lagoudas

**Center for Mechanics of Composites
Aerospace Engineering Department
Texas A&M University
College Station, TX 77843 - 3141**

* **Currently at The Department of Mechanical Engineering
University of Illinois at Chicago
M/C 251
2039 Engineering Research Facility
842 West Taylor Street
Chicago, Illinois 60607-7022**

Abstract

The phenomenological SMA equations developed in Part I are used in this second paper to derive the free energy and dissipation of a SMA composite material. The derivation consists of solving a boundary value problem formulated over a mesoscale representative volume element, followed by an averaging procedure to obtain the macroscopic composite constitutive equations. Explicit equations are derived for the transformation tensors that relate the composite transformation strain rate to the phase transformation rate in the fiber and matrix. Some key findings for the two-way SME in a SMA fiber/elastomer matrix composite are that processing-induced residual stresses alter the composite austenite start and martensite start temperatures, as well as the amount of composite strain recovered during a complete cycle of temperature and fiber martensite volume fraction. Relative to the two-way SME response of stiff-matrix composites, it was found that compliant-matrix composites: (1) Complete the phase transformation over a narrower temperature range; (2) Exhibit greater transformation strain during the reverse transformation; and (3) Undergo an incomplete strain cycle during a complete cycle of temperature and fiber martensite volume fraction. Due to the interaction of the fiber and matrix during transformation, macroscopic proportional stressing of the composite results in non-proportional fiber stressing, which in turn causes a small amount of martensitic reorientation to occur simultaneously with the transformation.

LIST OF SYMBOLS

Macroscopic Variables

The macroscopic variables are functions of $\{x_i, t\}$.

$\bar{\rho}$ = density

\hat{G} = Gibb's energy per unit mass
 \hat{s} = entropy per unit mass
 \hat{u} = internal energy per unit mass
 T = absolute temperature
 $\hat{\eta}$ = entropy produced due to microstructural rearrangements
 Σ_{ij} = Cuachy stress tensor
 E_{ij} = total strain tensor
 E_{ij}^D = total strain of deformation
 E_{ij}^{De} = elastic strain of deformation
 E_{ij}^{Dth} = thermal strain of deformation
 E_{ij}^{Dirr} = irreversible strain of deformation
 E_{ij}^{Dt} = the transformation strain of deformation
 E_{ij}^{Dr} = reorientation strain of deformation
 A_{ijkl}^1 = elastic compliance tensor
 A_{ij}^2 = thermal expansion vector
 A^3 = specific heat
 c^f = fiber volume fraction
 c^m = matrix volume fraction
 c^{Mf} = average martensite volume fraction in the fiber
 c^{Mm} = average martensite volume fraction in the matrix
 ϵ_{ij}^{irrf} = average irreversible strain in the fiber
 ϵ_{ij}^{irrm} = average irreversible strain in the matrix

Mesoscopic Variables

The mesoscopic variables are functions of $\{y_i, t\}$. The mesoscopic variables in the **deformed** state are

G = Gibb's energy per unit mass
 s = entropy per unit mass
 u = internal energy per unit mass
 σ_{ij} = Cauchy stress tensor
 c^M = martensite volume fraction
 σ_{ij}^E = stress that is in equilibrium with the applied tranctions
 σ_{ij}^{Rth} = residual (eigenstress) due to the thermal eigenstrain
 σ_{ij}^{Rirr} = residual (eigenstress) due to the irreversible eigenstrain
 ϵ_{ij} = total strain tensor
 ϵ_{ij}^e = elastic strain
 ϵ_{ij}^{te} = thermoelastic, or reversible, strain
 ϵ_{ij}^{th} = thermal strain
 ϵ_{ij}^{irr} = irreversible strain
 ϵ_{ij}^D = total strain of deformation
 ϵ_{ij}^{De} = elastic strain of deformation
 ϵ_{ij}^{Dth} = thermal strain of deformation

ϵ_{ij}^{Dirr} = irreversible strain of deformation
 $\tilde{\epsilon}_{ij}^{th}$ = kinematically admissible thermal strain
 $\tilde{\epsilon}_{ij}^{irr}$ = kinematically admissible irreversible strain
 u_i^D = displacement of deformation
 r = heat source, or sink, per unit volume
 \dot{Q} = heat rate per unit mass
 q_i = heat flux vector per unit volume
 B_{ijkl} = elastic stress localization tensor
 b_{ij} = thermal stress localization vector

Mesoscopic Parameters

The mesoscopic material parameters are functions of $\{y_i\}$. The mesoscopic material parameters in the **deformed** state are

ρ = density
 $a_{ijkl}^{A1}, a_{ijkl}^{M1}$ = austenite and martensite elastic compliance tensors
 a_{ij}^{A2}, a_{ij}^{M2} = austenite and martensite thermal expansion vectors
 a^{A3}, a^{M3} = austenite and martensite specific heats
 a^{A4}, a^{M4} = reference values of the austenite and martensite entropies
 a^{A5}, a^{M5} = reference values of the austenite and martensite Gibb's energies
 b^4 = kinematic hardening modulus
 b^7 = isotropic hardening modulus
 T^0 = absolute reference temperature

The mesoscopic material parameters in the **initial** state and the **manufactured** state are denoted by superscripts “ \oplus ” and “ $*$ ”, respectively:

T^* = manufacturing temperature
 ρ^\oplus, ρ^* = density
 G^\oplus, G^* = Gibb's energy per unit mass
 s^\oplus, s^* = entropy per unit mass
 u^\oplus, u^* = internal energy per unit mass
 σ_{ij}^* = Cauchy stress tensor
 $c^{M\oplus}, c^{M^*}$ = martensite volume fraction
 $\epsilon^\oplus, \epsilon_{ij}^*$ = total strain tensor
 ϵ_{ij}^{e*} = elastic strain
 $\epsilon^{th\oplus}, \epsilon_{ij}^{th*}$ = thermal strain
 $\epsilon^{irr\oplus}, \epsilon_{ij}^{irr*}$ = irreversible strain
 $a_{ijkl}^{1\oplus}, a_{ijkl}^{1*}$ = elastic compliance tensor
 $a_{ij}^{2\oplus}, a_{ij}^{2*}$ = thermal expansion vector
 $a^{3\oplus}, a^{3*}$ = specific heat
 $a^{4\oplus}, a^{4*}$ = reference value of the entropy
 $a^{5\oplus}, a^{5*}$ = reference value of the Gibb's energy

I. INTRODUCTION

The shape memory effect (SME) in monolithic shape memory alloy (SMA) materials can be due to the transformation between austenite and martensite and/or the reorientation of martensite variants. In the one-way shape memory effect, an external loading device must be used to bias the transformation and/or reorientation of martensite variants in order to create oriented martensite. When the oriented martensite is heated to make austenite, the inelastic strain is recovered. In the two-way shape memory effect (TWSME), internal (or residual) stresses within the monolithic SMA are used to bias the transformation of martensite variants. Thus, with no externally applied force, cooling of the austenite produces a transformation strain that is hysteretically recovered during heating of the martensite. The residual stresses necessary for the TWSME can be due to intrinsic defects such as dislocations, grain boundaries, etc. One-way SME materials can be “trained” to be TWSME materials by cyclically applying thermo-mechanical loads to create a favorable defect arrangement. Residual stresses in multiphase, or composite SMA materials can be due to the interaction of the various phases. Therefore, multiphase SMA materials can exhibit the TWSME even if no phase exhibits the TWSME due to intrinsic defects such as dislocations. (LAGOUDAS and TADJBAKHS [1992]; ESCHER and HORNBOKEN [1991,1992]; THUMANN et al [1992]). This composite TWSME has recently been studied as a means of making active structures. Also, it has been experimentally and analytically demonstrated that SMA fibers in a non-SMA metal matrix can be used to enhance the composite yield strength and damping capacity (FURUYA et al [1993]; YAMADA et al [1993]).

MURA [1982] has provided a general discussion of the linear elastic analysis of heterogeneous materials. A common reference in this work is the classical analysis of ESHELBY [1957], in which he determined the elastic fields inside an ellipsoidal inhomogeneity in an infinitely extended material. The MORI-TANKA [1973] micromechanics method, discussed herein in section III.2., is an approximate extension of Eshelby’s method to the case of interacting inhomogeneities. SUN et al [1991] applied the Mori-Tanaka method to the microscale problem of martensite inclusions in an austenite matrix in order to derive the free energy and evolution equations for the SME and pseudoelasticity due to phase transformation under nonproportional loading. The austenite and martensite were modelled as linear elastic, with identical elastic moduli. The transformation strain of the moving phase boundaries was approximated by allowing the volume fraction of martensite to evolve. More recently, RANIECKI and LEXCELLENT [1994] have provided general equations for the free energy of two phase linear elastic materials with misfitting inhomogeneities. These general relations were used to propose specific phenomenological constitutive equations for pseudoelasticity due to phase transformation. PATOOR et al [1991] studied the TWSME by formulating a microscale boundary value problem that included intrinsic defects, such as dislocations, and martensite/austenite boundaries. It was demonstrated that intrinsic defects create residual stresses that can drive the motion of martensite/austenite boundaries.

In section IV of Part I of this two-part paper, thermodynamically based phenomenological constitutive equations were proposed for a mesoscopic (Figure 1) material point. These equations are a combination and extension of the models of BERVEILLER et al [1991], ORTIN and PLANES [1988,1989,1991], PATOOR et al [1987,1993], RANIECKI and LEXCELLENT [1994] and SUN and HWANG [1993a,1993b] that account for the SME and pseudoelasticity with nonproportional loading, simultaneous transformation and reorientation, adiabatic deformation, and combined

isotropic and kinematic hardening. The TWSME was not included.

In section II of this second paper, the form of the constitutive equations and the material constants for the point R^{macro} in V^{macro} will be mesomechanically derived by first formulating a boundary value problem over V^{meso} , and then using an averaging procedure to obtain the constitution of point R^{macro} . The effects of processing-induced residual stresses are included in the mesomechanical analysis. The microscale interfacial motion is modelled using the rate-independent phenomenological constitutive equations of section IV of the first paper, Part I. The macroscale constitutive equations are derived using the four-step procedure recommended by SUQUET [1985,1987]: (1) Identify the representative volume element (RVE); (2) Define the macroscopic quantities; (3) Perform the localization; and (4) Perform the homogenization. The representative thermomechanical response of a SMA fiber/elastomer matrix composite is presented in section III.

Throughout the paper, superscripts denote qualitative description of the associated variable or constant, whereas subscripts denote tensorial components. There is no summation on repeated superscripts unless specifically stated by a summation symbol, \sum . A function \mathcal{H} of variables $(\mathcal{T}^1, \mathcal{T}^2, \dots)$ is denoted by brackets as $\mathcal{H}\{\mathcal{T}^1, \mathcal{T}^2, \dots\}$.

II. MICROTHERMODYNAMICS OF SHAPE MEMORY COMPOSITES

II.1. Representative Volume Element (RVE)

When viewed under a microscope, a macroscopic material point of a shape memory composite is seen to be a heterogeneous structure whose average properties over V^{meso} (Figure 1) are equal to the properties of the macroscopic material. The volume average over V^{meso} of any field variable $\mathcal{H}\{y\}$ is denoted by an overbar:

$$\overline{\mathcal{H}} = \frac{1}{V^{meso}} \int_{V^{meso}} \mathcal{H}\{y_i\} dV. \quad (1)$$

For a two phase composite with fiber and matrix phases, the volume average operator reduces to

$$\overline{\mathcal{H}} = \frac{1}{V^{meso}} \int_{V^{meso}} \mathcal{H}\{y_i\} dV = c^f \mathcal{H}^f + c^m \mathcal{H}^m, \quad (2)$$

where c^v is the volume fraction of the fiber ($v = f$) or matrix ($v = m$), and \mathcal{H}^v is the average of \mathcal{H} over the volume V^v of each phase:

$$\mathcal{H}^v = \frac{1}{V^v} \int_{V^v} \mathcal{H}\{y_i\} dV, \quad v = f, m. \quad (3)$$

The initial state at every point in V^{meso} is introduced to account for the initial manufacturing conditions. The initial internal energy is given by

$$u^\oplus = G^\oplus + T^\oplus s^\oplus, \quad (4)$$

where

$$G^\oplus = -\frac{1}{2} \frac{1}{\rho^{oplus}} a_{ijkl}^{1\oplus} \sigma_{ij}^\oplus \sigma_{kl}^\oplus - \frac{1}{\rho^{oplus}} a_{ij}^{2\oplus} \sigma_{ij}^\oplus (T^\oplus - T^\circ) + a^{3\oplus} \left[(T^\oplus - T^\circ) - T^\oplus \ln \left(\frac{T^\oplus}{T^\circ} \right) \right] - a^{4\oplus} T^\oplus + a^{5\oplus} + \frac{1}{2} b^7 (c^{M\oplus})^2 + \frac{1}{2} b^4 \epsilon_{ij}^{irr\oplus} \epsilon_{ij}^{irr\oplus}, \quad (5)$$

The initial strain at every point in V^{meso} is

$$\epsilon_{ij}^\oplus = \epsilon_{ij}^{e\oplus} + \epsilon_{ij}^{th\oplus} + \epsilon_{ij}^{irr\oplus} \quad (6)$$

where

$$\epsilon_{ij}^{e\oplus} = a_{ijkl}^{1\oplus} \sigma_{kl}^\oplus, \quad \epsilon_{ij}^{th\oplus} = a_{ij}^{2\oplus} (T^\oplus - T^\circ), \quad (7a - b)$$

and $\epsilon_{ij}^{irr\oplus}$ can be determined by averaging the kinematics of the microstructural rearrangements over V^{micro} . The initial stress, σ_{ij}^\oplus , is a residual, or self-equilibrated stress:

$$\overline{\sigma_{ij}^\oplus} = 0. \quad (8)$$

The phases may be separated in the initial state. The phases are subjected to thermomechanical loading during the composite processing, which takes the initial state to the manufactured state. The manufactured state at every point in V^{meso} is characterized by

$$u^* = G^* + T^* s^*, \quad (9)$$

where

$$G^* = -\frac{1}{2} \frac{1}{\rho} a_{ijkl}^{1*} \sigma_{ij}^* \sigma_{kl}^* - \frac{1}{\rho} a_{ij}^{2*} \sigma_{ij}^* (T^* - T^\circ) + a^{3*} \left[(T^* - T^\circ) - T^* \ln \left(\frac{T^*}{T^\circ} \right) \right] - a^{4*} T^* + a^{5*} + \frac{1}{2} b^7 (c^{M*})^2 + \frac{1}{2} b^4 \epsilon_{ij}^{irr*} \epsilon_{ij}^{irr*}, \quad (10)$$

and the internal energy of processing, u^P , is given by

$$u^P = u^* - u^\oplus = G^P + T^* s^* - T^\oplus s^\oplus, \quad G^P = G^* - G^\oplus. \quad (11a - b)$$

It is assumed that T^* is homogeneous, i.e. $T_{,i}^* = 0$, where the differentiation is taken with respect to the mesoscale (y_i) coordinates. This limitation requires that all phases are brought together at the same temperature. σ_{ij}^* is a residual, or self-equilibrated stress:

$$\overline{\sigma_{ij}^*} = 0. \quad (12)$$

The strain at every point in V^{meso} in the manufactured state is

$$\epsilon_{ij}^* = \epsilon_{ij}^{e*} + \epsilon_{ij}^{th*} + \epsilon_{ij}^{irr*} = \epsilon_{ij}^P + \epsilon_{ij}^\oplus = \epsilon_{ij}^{Pe} + \epsilon_{ij}^{Pth} + \epsilon_{ij}^{Pirr} + \epsilon_{ij}^\oplus, \quad (13)$$

where

$$\begin{aligned}\epsilon_{ij}^{e*} &= a_{ijkl}^{1*} \sigma_{kl}^* = \epsilon_{ij}^{Pe} + \epsilon_{ij}^{e\oplus} = \epsilon_{ij}^{Pe} + a_{ijkl}^{1\oplus} \sigma_{kl}^{\oplus}, & \epsilon_{ij}^{irr*} &= \epsilon_{ij}^{Pirr} + \epsilon_{ij}^{irr\oplus}, \\ \epsilon_{ij}^{th*} &= a_{ij}^{2*} (T^* - T^0) = \epsilon_{ij}^{Pth} + a_{ij}^{2\oplus} (T^{\oplus} - T^0).\end{aligned}\quad (14a - c)$$

The manufactured stress, σ_{ij}^* , could be nonzero even if $\epsilon_{ij}^{irr*} = \epsilon_{ij}^{th*} = 0$. For example, if the fibers were elastically stressed by an external loading device, and the matrix was then affixed to the fibers, and the external loads were then removed from the fibers, a nonzero σ_{ij}^* would result if a_{ijkl}^{1*} is not spatially homogeneous, even if $\epsilon_{ij}^{irr*} = \epsilon_{ij}^{th*} = 0$. The manufactured state is transformed into the deformed state by the kinematically admissible displacements u_i^D . The deformed state at every point in V^{meso} is characterized by

$$u = G + Ts, \quad (15)$$

where

$$\begin{aligned}G &= -\frac{1}{2} \frac{1}{\rho} a_{ijkl}^1 \sigma_{ij} \sigma_{kl} - \frac{1}{\rho} a_{ij}^2 \sigma_{ij} \Delta T + a^3 \left[\Delta T - T \ln \left(\frac{T}{T^0} \right) \right] - a^4 T + a^5 \\ &\quad + \frac{1}{2} b^7 (c^M)^2 + \frac{1}{2} b^4 \epsilon_{ij}^{irr} \epsilon_{ij}^{irr}, \quad \Delta T = T - T^0,\end{aligned}\quad (16a - b)$$

and the internal energy of deformation, u^D , is given by

$$u^D = u - u^* = G^D + Ts - T^* s^*, \quad G^D = G - G^*. \quad (17a - b)$$

If the deformation is isothermal, i.e. $T = T^*$, then it is possible to write u^D in terms of the entropy of deformation, s^D :

$$u^D = G^D + T^* ds^D, \quad s^D = s - s^*. \quad (18a - b)$$

The strain in the deformed state is

$$\epsilon_{ij} = \epsilon_{ij}^e + \epsilon_{ij}^{th} + \epsilon_{ij}^{irr} = \frac{1}{2} (u_{i,j}^D + u_{j,i}^D) + \epsilon_{ij}^* = \epsilon_{ij}^D + \epsilon_{ij}^* = \epsilon_{ij}^{De} + \epsilon_{ij}^{Dth} + \epsilon_{ij}^{Dirr} + \epsilon_{ij}^*, \quad (19)$$

from which it may be seen that

$$\begin{aligned}\epsilon_{ij}^{De} &= \epsilon_{ij}^e - \epsilon_{ij}^{e*} = a_{ijkl}^1 \sigma_{kl} - a_{ijkl}^{1*} \sigma_{kl}^* & \epsilon_{ij}^{Dirr} &= \epsilon_{ij}^{irr} - \epsilon_{ij}^{irr*}, \\ \epsilon_{ij}^{Dth} &= \epsilon_{ij}^{th} - \epsilon_{ij}^{th*} = a_{ij}^2 (T - T^0) - a_{ij}^{2*} (T^* - T^0).\end{aligned}\quad (20a - c)$$

Also, using (13),

$$\epsilon_{ij}^D = \epsilon_{ij} - \epsilon_{ij}^* = \frac{1}{2} (u_{i,j}^D + u_{j,i}^D). \quad (21)$$

The deformation is assumed to be quasistatic with no body forces, and the equilibrium equations are given by

$$\sigma_{ij,i} = 0 \quad \text{in } V^{meso}. \quad (22)$$

It is assumed that the temperature is homogeneous within V^{meso} , i.e.

$$T_{,i} = 0 \quad (23)$$

where the derivative is taken with respect to the mesoscale (y_i) coordinates. The conservation of energy (equations (78) and (79) of the first paper) is given by

$$\rho \dot{u} = \sigma_{ij} \dot{\epsilon}_{ij} + \rho \dot{Q}. \quad (24)$$

and it is assumed that the heat flux, q_i , vanishes:

$$q_i = 0, \quad \rightarrow \quad \dot{Q} = r - q_{i,i} = r. \quad (25)$$

This assumption of local thermal equilibrium allows for the definition of mesoscopic and macroscopic entropy as in themostatics. Large thermal gradients within V^{meso} can cause mesoscopic thermoelastic damping that manifests itself through macroscopic viscoelasticity (MOLINARI and ORTIZ [1987]).

II.2. Definition of the Macroscopic Variables

The macroscopic extrinsic variables such as free energy, \hat{G} , internal energy, \hat{u} , entropy, \hat{s} , and dissipation $T\hat{\eta}$ are governed by (using (23))

$$\bar{\rho} \hat{G} = \overline{\rho G}, \quad \bar{\rho} \hat{u} = \overline{\rho u}, \quad \bar{\rho} \hat{s} = \overline{\rho s}, \quad T\hat{\eta} = \overline{T\eta} = T\bar{\eta}. \quad (26a - d)$$

The macroscopic stress Σ_{ij} is given by

$$\Sigma_{ij} = \overline{\sigma_{ij}} = \frac{1}{V^{meso}} \int_{\partial V^{meso}} \frac{1}{2} (\sigma_{ik} y_j n_k + \sigma_{jk} y_i n_k) ds = \frac{1}{V^{meso}} \int_{\partial V^{meso}} \frac{1}{2} (t_i y_j + t_j y_i) ds, \quad (27)$$

where t_i is the traction vector. From (21) and (33), it follows that the macroscopic stress vanishes in the manufactured state:

$$\Sigma_{ij}^* = 0. \quad (28)$$

The macroscopic strain of deformation, E_{ij}^D , is

$$E_{ij}^D = E_{ij} - E_{ij}^* = \frac{1}{V^{meso}} \int_{\partial V^{meso}} \frac{1}{2} (u_i^D n_j + u_j^D n_i) ds = \overline{\epsilon_{ij}^D}, \quad E_{ij} = \overline{\epsilon_{ij}}, \quad E_{ij}^* = \overline{\epsilon_{ij}^*}. \quad (29a - c)$$

II.3. Localization Procedure

Stress-free strains such as ϵ_{ij}^{th} and ϵ_{ij}^{irr} are sometimes known as eigenstrains (MURA [1982]). The mesomechanical derivation of the macroscopic eigenstrains is complicated by the fact that they are not, in general, equal to the volume average of the corresponding mesoscopic eigenstrains. An eigenstrain does not volume average if it does not derive from a continuous displacement field. In general,

$$E_{ij}^{De} \neq \overline{\epsilon_{ij}^{De}}, \quad E_{ij}^{Dth} \neq \overline{\epsilon_{ij}^{Dth}}, \quad E_{ij}^{Dirr} \neq \overline{\epsilon_{ij}^{Dirr}}. \quad (30a - c)$$

In order to derive equations for the macroscopic eigenstrains, it is necessary to perform a localization process in which the mesoscopic fields are derived from the boundary conditions on ∂V^{meso} . (SUQUET [1985, 1987]). The three most commonly used boundary conditions are: (1) Uniform stresses on ∂V^{meso} ; (2) Uniform strains on ∂V^{meso} ; and (3) Periodicity conditions. The uniform stress condition is used herein:

$$\sigma_{ij}n_j = \Sigma_{ij}n_j \quad \text{on} \quad \partial V^{meso}. \quad (31)$$

For the isothermal elastic response, i.e. $\dot{T} = \dot{c}^M = \dot{\epsilon}_{ij}^{irr} = 0$, the solution to the mesoscopic boundary value problem yields the elastic stress localization tensor B_{ijkl} , which is the meso-to-macro map

$$\sigma_{ij}^E = B_{ijkl}\Sigma_{kl}, \quad \overline{B_{ijkl}} = I_{ijkl}. \quad (32a - b)$$

In order to maintain, in the macroscopic space, the definition of eigenstrain as a stress-free strain, the macroscopic strains must be related to the macroscopic work using the Hill-Mandel macrohomogeneity condition, a specialized form of virtual work. As a first step in deriving equations for the macroscopic eigenstrains, the total macroscopic deformation strain E_{ij}^D is decomposed as

$$E_{ij}^D = E_{ij}^{Dte} + E_{ij}^{Dirr} = E_{ij}^{De} + E_{ij}^{Dth} + E_{ij}^{Dirr}, \quad E_{ij}^{Dirr} = E_{ij}^{Dt} + E_{ij}^{Dr}, \quad (33a - b)$$

where E_{ij}^{Dt} and E_{ij}^{Dr} , are the macroscopic deformational transformation strain and deformational reorientation strain. The Hill-Mandel macrohomogeneity condition, a specialized form of virtual work, is written as

$$\overline{\sigma_{ij}^\bullet \epsilon_{ij}^\bullet} = \overline{\Sigma_{ij}^\bullet E_{ij}^\bullet}, \quad (34)$$

where σ_{ij}^\bullet and ϵ_{ij}^\bullet are arbitrary statically admissible stress and kinematically admissible strain fields. A special form of the macrohomogeneity condition that is useful in the following derivation is given by

$$0 = \overline{\sigma_{ij}^{SE} \epsilon_{ij}^{KA}} = \overline{\Sigma_{ij}^{SA} \epsilon_{ij}^{PF}}, \quad (35)$$

where “SE”, “KA”, “SA” and “PF” indicate self-equilibrated, kinematically admissible, statically admissible and purely fluctuating, respectively. The stress is decomposed into a part σ_{ij}^E that is in equilibrium with the tractions on ∂V^{meso} and a part σ_{ij}^R that is residual (self-equilibrated):

$$\sigma_{ij} = \sigma_{ij}^E + \sigma_{ij}^R, \quad \sigma_{ij}^R = \sigma_{ij}^{Rirr} + \sigma_{ij}^{Rth} + \sigma_{ij}^*, \quad (36a - b)$$

and

$$\overline{\sigma_{ij}^E} = \overline{\sigma_{ij}} = \Sigma_{ij}, \quad \overline{\sigma_{ij}^R} = \overline{\sigma_{ij}^{Rirr}} = \overline{\sigma_{ij}^{Rth}} = \overline{\sigma_{ij}^*} = 0. \quad (37a - b)$$

The residual stresses in (36) and (37) are called internal stresses within the materials engineering community and eigenstresses within the micromechanics community. The eigenstresses σ_{ij}^{Rth} and σ_{ij}^{Rirr} are due to the incompatibility of the eigenstrains ϵ_{ij}^{Dth} and ϵ_{ij}^{Dirr} , respectively. In the manufactured state, $\sigma_{ij}^{Rth} = \sigma_{ij}^{Rirr} = 0$. In order to apply the macrohomogeneity condition, it is first necessary to define modified eigenstrains, $\tilde{\epsilon}_{ij}^{eig}$, that are kinematically admissible:

$$\begin{aligned} \tilde{\epsilon}_{ij}^{th} &= a_{ijkl}^1 \sigma_{kl}^{Rth} + \epsilon_{ij}^{Dth} = a_{ijkl}^1 \sigma_{kl}^{Rth} + a_{ij}^2 \Delta T - a_{ij}^{2*} (T^* - T^0), \\ \tilde{\epsilon}_{ij}^{irr} &= a_{ijkl}^1 \sigma_{kl}^{Rirr} + \epsilon_{ij}^{Dirr} = a_{ijkl}^1 \sigma_{kl}^{Rirr} + (\epsilon_{ij}^{irr} - \epsilon_{ij}^{irr*}). \end{aligned} \quad (38a - b)$$

The total deformation strain can now be written as the sum of three kinematically admissible strains:

$$\epsilon_{ij}^D = \epsilon_{ij} - \epsilon_{ij}^* = \epsilon_{ij}^E + \tilde{\epsilon}_{ij}^{th} + \tilde{\epsilon}_{ij}^{irr}, \quad (39)$$

where the strain ϵ_{ij}^E is associated by Hooke's law with the stress σ_{ij}^E as

$$\epsilon_{ij}^E = a_{ijkl}^1 \sigma_{kl}^E. \quad (40)$$

Note that ϵ_{ij}^E is kinematically admissible because it is the total strain that results from an isothermal elastic input E_{ij}^{De} . Application of the macrohomogeneity condition results in

$$\begin{aligned} \overline{\sigma_{ij}^\bullet \epsilon_{ij}^E} &= \Sigma_{ij}^\bullet E_{ij}^{De}, & \overline{\sigma_{ij}^\bullet \tilde{\epsilon}_{ij}^{th}} &= \Sigma_{ij}^\bullet E_{ij}^{Dth}, & \overline{\sigma_{ij}^\bullet \tilde{\epsilon}_{ij}^{irr}} &= \Sigma_{ij}^\bullet E_{ij}^{Dirr}, \\ \overline{\sigma_{ij}^\bullet \epsilon_{ij}^D} &= \overline{\sigma_{ij}^\bullet (\epsilon_{ij} - \epsilon_{ij}^*)} = \Sigma_{ij}^\bullet E_{ij}^D. \end{aligned} \quad (41a-d)$$

For the special cases in which $a_{ij}^{2*} = a_{ij}^2$, or $T^\circ = T^*$, it is possible to obtain the thermal stress localization tensor b_{ij} as

$$\sigma_{ij}^{Rth} = b_{ij} \Delta T^*, \quad \overline{b_{ij}} = 0, \quad \Delta T^* = T - T^*. \quad (42a-c)$$

Because T° is the mesoscopic reference temperature, and not a constitutive property, it is always possible to set $T^\circ = T^*$, in which case

$$T^\circ = T^*, \quad \rightarrow \quad \Delta T = \Delta T^*, \quad \rightarrow \quad \sigma_{ij}^{Rth} = b_{ij} \Delta T = b_{ij} \Delta T^*, \quad (43)$$

and T° and ΔT are now spatially homogeneous in V^{meso} . Although (43) simplifies the micromechanics problem, additional work may be required to recalculate some of the material constants in G^\oplus , G^* and G if T° is changed to T^* .

For a two phase composite in which the material properties, eigenstrains, and state variables are assumed to be uniform within each phase, it is possible to determine the eigenstresses as a function of the eigenstrains, the stress concentration tensors, and the constituent mechanical properties (BENVENISTE and DVORAK [1990]). The irreversible eigenstress is

$$\sigma_{ij}^{Rirr v} = \mathcal{P}_{ijmn}^v [(e_{mn}^{irr f} - \epsilon_{mn}^{irr *f}) - (\epsilon_{mn}^{irr m} - \epsilon_{mn}^{irr *m})], \quad v = f, m, \quad (44)$$

where

$$\mathcal{P}_{ijmn}^v = (I_{ijkl} - B_{ijkl}^v) (a_{klmn}^{1m} - a_{klmn}^{1f})^{-1}, \quad v = f, m. \quad (45)$$

Similarly, for a two phase composite in which T° is also uniform within each phase, the thermal eigenstress is

$$\sigma_{ij}^{Rth v} = \mathcal{P}_{ijmn}^v \left\{ [a_{mn}^{2f} (T - T^{f^\circ}) - a_{mn}^{2*f} (T^* - T^{f^\circ})] - [a_{mn}^{2m} (T - T^{m^\circ}) - a_{mn}^{2*m} (T^* - T^{m^\circ})] \right\}. \quad (46)$$

It is not possible to factor the term ΔT^* in σ_{ij}^{Rthv} unless $a_{ij}^{2*v} = a_{ij}^{2v}$ or $T^{f\circ} = T^{m\circ} = T^\circ = T^*$. When (43) applies, (46) can be written as

$$\sigma_{ij}^{Rthv} = b_{ij}^v \Delta T^* = b_{ij}^v \Delta T, \quad b_{ij}^v = \mathcal{P}_{ijmn}^v (a_{mn}^{2f} - a_{mn}^{2m}), \quad v = f, m. \quad (47a - b)$$

This decomposition scheme can be extended to multi-phase materials in which the fields are assumed to be uniform within each phase (DVORAK and BENVENISTE [1992]).

II.4. Homogenization

II.4.1. Homogenization of the Strains

The mesoscopic Hooke's law (40) and (32a) yield

$$\epsilon_{ij}^E = a_{ijkl}^1 \sigma_{kl}^E = a_{ijkl}^1 B_{klmn} \Sigma_{mn}. \quad (48)$$

The macroscopic elastic strain can be derived by using the statically admissible stress $\sigma_{ij}^\bullet = \sigma_{ij}^E$ and (32a) and (48) in (41a) to obtain

$$E_{kl}^{De} = \overline{B_{ijkl} a_{ijop}^1 B_{opmn}} \Sigma_{mn}, \quad (49)$$

from which it may be seen that the macroscopic elastic compliance A_{klmn}^1 is

$$A_{klmn}^1 = \overline{B_{ijkl} a_{ijop}^1 B_{opmn}}. \quad (50)$$

The macroscopic elastic modulus could also be determined from (40):

$$E_{kl}^{De} = \overline{\epsilon_{kl}^E} = \overline{a_{kl ij}^1 B_{ijmn}} \Sigma_{mn}, \quad (51)$$

which yields an alternative equation for A_{klmn}^1 :

$$A_{klmn}^1 = \overline{a_{kl ij}^1 B_{ijmn}}. \quad (52)$$

The macroscopic eigenstrains can be derived by using the statically admissible stress $\sigma_{ij}^\bullet = \sigma_{ij}^E$ given by (32) and the K.A. strains (38) in (41b-c) to obtain

$$\begin{aligned} \Sigma_{kl} E_{kl}^{Dth} &= \Sigma_{kl} \left[\overline{B_{ijkl} [a_{ij}^2 \Delta T - a_{ij}^{2*} (T^* - T^\circ)]} + \overline{B_{ijkl} a_{ijmn}^1 \sigma_{mn}^{Rth}} \right], \\ \Sigma_{kl} E_{kl}^{Dirr} &= \Sigma_{kl} \left[\overline{B_{ijkl} (\epsilon_{ij}^{irr} - \epsilon_{ij}^{irr*})} + \overline{B_{ijkl} a_{ijmn}^1 \sigma_{mn}^{Rirr}} \right]. \end{aligned} \quad (53a - b)$$

Equations (53) can be simplified by using (35) to eliminate the volume average of the product of the kinematically admissible strain $\Sigma_{kl} B_{ijkl} a_{ijmn}^1$ and the self-equilibrated stresses, thereby leading to the following expressions for the global eigenstrains:

$$E_{kl}^{Dth} = \overline{B_{ijkl} [a_{ij}^2 \Delta T - a_{ij}^{2*} (T^* - T^\circ)]}, \quad E_{kl}^{Dirr} = \overline{B_{ijkl} (\epsilon_{ij}^{irr} - \epsilon_{ij}^{irr*})}, \quad (54a - b)$$

from which it may be seen that the macroscopic thermal expansion tensor, A_{kl}^2 is

$$A_{kl}^2 = \overline{B_{ijkl}a_{ij}^2}. \quad (55)$$

II.4.2. Homogenization of the Free Energy

The total macroscopic free energy per unit total mass, \hat{G} , is given by

$$\bar{\rho}\hat{G} = \overline{\rho G} = \overline{\rho G^e} + \overline{\rho G^{th}} + \overline{\rho G^{eig}}, \quad (56)$$

where, from (52)-(58) and (125) of the first paper, the SMA Gibb's energy for each point in V^{meso} is

$$\begin{aligned} \rho G^e &= \rho G^e\{\sigma_{ij}, T, c^M\} = -\frac{1}{2}a_{ijkl}^1\sigma_{ij}\sigma_{kl} - a_{ij}^2\sigma_{ij}\Delta T, \\ \rho G^{th} &= \rho G^{th}\{T, c^M\} = \rho a^3 \left[\Delta T - T \ln\left(\frac{T}{T^0}\right) \right] - \rho a^4 T + \rho a^5, \\ \rho G^{eig} &= \rho G^{eig}\{c^M, \epsilon_{ij}^{irr}\} = \frac{1}{2}\rho b^7 (c^M)^2 + \frac{1}{2}\rho b^4 \epsilon_{ij}^{irr} \epsilon_{ij}^{irr}, \end{aligned} \quad (57a-c)$$

and $G^{eig} = 0$ for points that correspond to an elastic material. The mesoscopic elastic free energy is averaged first:

$$\begin{aligned} \overline{\rho G^e} &= -\frac{1}{2}\overline{a_{ijkl}^1\sigma_{ij}\sigma_{kl} - a_{ij}^2\sigma_{ij}\Delta T} \\ &= -\frac{1}{2}\overline{a_{ijkl}^1(B_{ijmn}\Sigma_{mn} + \sigma_{ij}^R)(B_{kl\sigma p}\Sigma_{\sigma p} + \sigma_{kl}^R)} - \overline{a_{ij}^2(B_{ij\sigma p}\Sigma_{\sigma p} + \sigma_{ij}^R)\Delta T}. \end{aligned} \quad (58)$$

Because $a_{ijkl}^1 B_{ijmn} \Sigma_{mn}$ is K.A. and σ_{ij}^R is S.E., application of (35) leads to the simplified form

$$\begin{aligned} \overline{\rho G^e} &= -\frac{1}{2}A_{mnop}^1 \Sigma_{mn} \Sigma_{\sigma p} - A_{\sigma p}^2 \Sigma_{\sigma p} T + \overline{a_{ij}^2 B_{ij\sigma p} T^0 \Sigma_{\sigma p}} \\ &\quad - \frac{1}{2}\overline{a_{ijkl}^1 (\sigma_{ij}^{Rirr} + \sigma_{ij}^{Rth} + \sigma_{ij}^*) (\sigma_{kl}^{Rirr} + \sigma_{kl}^{Rth} + \sigma_{kl}^*)} - \overline{a_{ij}^2 (\sigma_{ij}^{Rirr} + \sigma_{ij}^{Rth} + \sigma_{ij}^*) \Delta T}. \end{aligned} \quad (59)$$

The eigenstrains can be explicitly included in (59) by combining (13), (14) and (38) to obtain

$$\begin{aligned} \sigma_{ij}^R &= (a_{ijkl}^1)^{-1} (\tilde{\epsilon}_{kl}^{irr} + \tilde{\epsilon}_{kl}^{th}) + (a_{ijkl}^{1*})^{-1} \epsilon_{kl}^* - (a_{ijkl}^1)^{-1} (\epsilon_{kl}^{irr} + a_{kl}^2 \Delta T) \\ &\quad + [(a_{ijkl}^1)^{-1} - (a_{ijkl}^{1*})^{-1}] [\epsilon_{kl}^{irr*} + a_{kl}^{2*} (T^* - T^0)]. \end{aligned} \quad (60)$$

For the case in which $a_{ijkl}^1 = a_{ijkl}^{1*}$, the last term in (60) vanishes, and (60) and (35) can be used to rewrite (59) by eliminating the volume average of the product of the S.E. stress σ_{ij}^R and the K.A. strains $\tilde{\epsilon}_{ij}^{th}$, $\tilde{\epsilon}_{ij}^{irr}$, and ϵ_{ij}^* , resulting in

$$\begin{aligned} \overline{\rho G^e} &= -\frac{1}{2}A_{mnop}^1 \Sigma_{mn} \Sigma_{\sigma p} - A_{\sigma p}^2 \Sigma_{\sigma p} T + \overline{a_{ij}^2 B_{ij\sigma p} T^0 \Sigma_{\sigma p}} \\ &\quad + \frac{1}{2}\overline{(\sigma_{ij}^{Rirr} + \sigma_{ij}^{Rth} + \sigma_{ij}^*) (\epsilon_{ij}^{irr} - a_{ij}^2 \Delta T)}. \end{aligned} \quad (61)$$

For the case in which $T^\circ = T^*$, (47) can be used to rewrite (59) as

$$\begin{aligned} \overline{\rho G^e} &= -\frac{1}{2}A_{mnop}^1 \Sigma_{mn} \Sigma_{op} - A_{op}^2 \Sigma_{op} \Delta T \\ &\quad - \frac{1}{2} \overline{a_{ijkl}^1 (\sigma_{ij}^{Rirr} + b_{ij} \Delta T + \sigma_{ij}^*) (\sigma_{kl}^{Rirr} + b_{kl} \Delta T + \sigma_{kl}^*)} - \overline{a_{ij}^2 (\sigma_{ij}^{Rirr} + b_{ij} \Delta T + \sigma_{ij}^*)} \Delta T. \end{aligned} \quad (62)$$

For $T^\circ = T^*$, the thermal eigenstrain can be eliminated by using (35) and (38a) to derive,

$$\begin{aligned} \overline{b_{ij} a_{ij}^2} (\Delta T)^2 &= \overline{\sigma_{ij}^{Rth} a_{ij}^2} \Delta T = \overline{\sigma_{ij}^{Rth} \epsilon_{ij}^{th}} = \overline{\sigma_{ij}^{Rth} (\tilde{\epsilon}_{ij}^{th} - a_{ijkl}^1 \sigma_{kl}^{Rth})} \\ &= -\overline{a_{ijkl}^1 \sigma_{ij}^{Rth} \sigma_{kl}^{Rth}} = -\overline{a_{ijkl}^1 b_{ij} b_{kl}} (\Delta T)^2, \end{aligned} \quad (63)$$

which can be used in (62) to obtain

$$\begin{aligned} \overline{\rho G^e} &= -\frac{1}{2}A_{mnop}^1 \Sigma_{mn} \Sigma_{op} - A_{op}^2 \Sigma_{op} \Delta T - \frac{1}{2} \overline{a_{ijkl}^1 (\sigma_{ij}^{Rirr} + \sigma_{ij}^*) (\sigma_{kl}^{Rirr} + \sigma_{kl}^*)} \\ &\quad - \overline{(\sigma_{ij}^{Rirr} + \sigma_{ij}^*) (a_{ijkl}^1 b_{kl} + a_{ij}^2)} \Delta T + \frac{1}{2} \overline{a_{ijkl}^1 b_{ij} b_{kl}} (\Delta T)^2. \end{aligned} \quad (64)$$

Equations (38a) and (35) can be used to eliminate term containing the K.A. strain $\tilde{\epsilon}_{ij}^{th}$, resulting in

$$\begin{aligned} \overline{\rho G^e} &= -\frac{1}{2}A_{mnop}^1 \Sigma_{mn} \Sigma_{op} - A_{op}^2 \Sigma_{op} \Delta T \\ &\quad - \frac{1}{2} \overline{a_{ijkl}^1 (\sigma_{ij}^{Rirr} + \sigma_{ij}^*) (\sigma_{kl}^{Rirr} + \sigma_{kl}^*)} + \frac{1}{2} \overline{a_{ijkl}^1 b_{ij} b_{kl}} (\Delta T)^2. \end{aligned} \quad (65)$$

In general, the volume averaged free energy contains coupling terms in the various eigenstresses. However, the coupling terms between the thermal and the non-thermal eigenstresses vanish because of the term in G containing $\sigma_{ij}^R a_{ij}^2 \Delta T$.

When (43) is applicable the macroscopic specific heat, A^3 , is

$$\bar{\rho} A^3 = -T \frac{\partial^2 \overline{\rho G}}{\partial T^2} = \overline{\rho a^3} - \left(\overline{a_{ijkl}^1 b_{ij} b_{kl}} \right) T. \quad (66)$$

The lattice excitation, represented by the specific heat a^3 , reduces the free energy with increasing temperature. However, the thermal eigenstresses contribute a positive semi-definite term that is quadratic in temperature. Thus, thermal eigenstresses appear to a mesoscopic observer as elastic energy, and to a macroscopic observer as purely thermal energy. The composite will become thermally unstable, i.e. $A^3 < 0$ at the temperature

$$T = \overline{\rho a^3} \left(\overline{a_{ijkl}^1 b_{ij} b_{kl}} \right)^{-1}. \quad (67)$$

For a two phase composite in which the material properties, eigenstrains, and state variables are assumed to be uniform within each phase, (44) and (46) can be used in (59) to decompose $\bar{\rho} \hat{G}$ as

$$\begin{aligned} \bar{\rho} \hat{G} \{ \Sigma_{ij}, T, c^{Mv}, \epsilon_{ij}^{irr v}, \sigma_{ij}^{*v} \} &= \bar{\rho} \hat{G}^e \{ \Sigma_{ij}, T, c^{Mv} \} + \bar{\rho} \hat{G}^{th} \{ T, c^{Mv}, \sigma_{ij}^{*v} \} \\ &\quad + \bar{\rho} \hat{G}^{eig} \{ T, c^{Mv}, \epsilon_{ij}^{irr v}, \sigma_{ij}^{*v} \}, \quad v = f, m. \end{aligned} \quad (68)$$

Furthermore, it will be shown that the free energies can be decomposed into a first part that is obtained by treating the composite phases, such as the fiber and matrix, as pure substances, for which there is no phase interaction, and a second part due to mixing of the phases, which accounts for the phase interaction:

$$\hat{G}^e = \hat{G}^{e\,pure} + \hat{G}^{e\,mix}, \quad \hat{G}^{th} = \hat{G}^{th\,pure} + \hat{G}^{th\,mix}, \quad \hat{G}^{eig} = \hat{G}^{eig\,pure} + \hat{G}^{eig\,mix}. \quad (69a - c)$$

The mixing energy includes a part due to manufacturing and a part due to deformation. In general,

$$\begin{aligned} \bar{\rho} \hat{G}^e \{\Sigma_{ij}, T, c^{Mv}\} &\neq \overline{\rho G^e}, & \bar{\rho} \hat{G}^{th} \{T, c^{Mv}, \sigma_{ij}^{*v}\} &\neq \overline{\rho G^{th}}, \\ \bar{\rho} \hat{G}^{eig} \{T, c^{Mv}, \epsilon_{ij}^{irr\,v}, \sigma_{ij}^{*v}\} &\neq \overline{\rho G^{eig}}, & v &= f, m. \end{aligned} \quad (70a - c)$$

Comparison of (56), (57) and (59) with (68) - (70) yields

$$\begin{aligned} \bar{\rho} \hat{G}^{e\,pure} \{\Sigma_{ij}, T, c^{Mv}\} &= -\frac{1}{2} \overline{a_{ijkl}^1} \Sigma_{ij} \Sigma_{kl} - \overline{a_{ij}^2} \Sigma_{ij} T + \overline{a_{ij}^2 T^\circ} \Sigma_{ij} \\ \bar{\rho} \hat{G}^{e\,mix} \{\Sigma_{ij}, T, c^{Mv}\} &= -\frac{1}{2} \overline{a_{ijmn}^1 (B_{mnkl} - I_{mnkl})} \Sigma_{ij} \Sigma_{kl} - \overline{a_{mn}^2 (B_{mnij} - I_{mnij})} \Sigma_{ij} T \\ &\quad + \overline{a_{mn}^2 (B_{mnij} - I_{mnij}) T^\circ} \Sigma_{ij}, \end{aligned} \quad (71a - b)$$

and

$$\begin{aligned} \bar{\rho} \hat{G}^{th\,pure} \{T, c^{Mv}\} &= \overline{\rho a^3 \left[\Delta T - T \ln \left(\frac{T}{T^\circ} \right) \right]} - \overline{\rho a^4 T} + \overline{\rho a^5} \\ \bar{\rho} \hat{G}^{th\,mix} \{T, c^{Mv}, \sigma_{ij}^{*v}\} &= -\frac{1}{2} \overline{a_{ijkl}^1 \sigma_{ij}^{Rth} \sigma_{kl}^{Rth}} - \overline{a_{ij}^2 (\sigma_{ij}^{Rth} + \sigma_{ij}^{*v}) \Delta T} - \overline{a_{ijkl}^1 \sigma_{ij}^{Rth} \sigma_{kl}^*}, \end{aligned} \quad (72a - b)$$

and

$$\begin{aligned} \bar{\rho} \hat{G}^{eig\,pure} \{c^{Mv}, \epsilon_{ij}^{irr\,v}\} &= \frac{1}{2} \overline{\rho b^7 (c^M)^2} + \frac{1}{2} \overline{\rho b^4 \epsilon_{ij}^{irr} \epsilon_{ij}^{irr}} \\ \bar{\rho} \hat{G}^{eig\,mix} \{T, c^{Mv}, \epsilon_{ij}^{irr\,v}, \sigma_{ij}^{*v}\} &= -\frac{1}{2} \overline{a_{ijkl}^1 \sigma_{ij}^{Rirr} (\sigma_{kl}^{Rirr} + 2\sigma_{kl}^{Rth} + 2\sigma_{kl}^*)} - \frac{1}{2} \overline{a_{ijkl}^1 \sigma_{ij}^* \sigma_{kl}^*} \\ &\quad - \overline{\sigma_{ij}^{Rirr} a_{ij}^2 \Delta T}. \end{aligned} \quad (73a - b)$$

The thermodynamic conjugates to the state variables that characterize the deformation can be derived using the macroscopic free energy, $\hat{G} = \hat{G}\{\Sigma_{ij}, T; c^M, \epsilon_{ij}^{irr}\}$, where c^M and ϵ_{ij}^{irr} are fields within V^{meso} . Specifically, equations (49) and (54a) can also be derived using

$$E_{ij}^{Dte} = E_{ij}^{De} + E_{ij}^{Dth} = -\bar{\rho} \left[\frac{\partial \hat{G}}{\partial \Sigma_{ij}} - \left(\frac{\partial \hat{G}}{\partial \Sigma_{ij}} \right)^* \right], \quad (74)$$

Similarly, the entropy of deformation can be derived using

$$\hat{s}^D = \hat{s} - \hat{s}^*, \quad \hat{s}^D = \frac{\partial \hat{G}}{\partial T}, \quad \hat{s}^* = \left(\frac{\partial \hat{G}}{\partial T} \right)^*. \quad (75a - c)$$

For a two-phase composite in which the material properties, eigenstrains, and state variables are assumed to be uniform within each phase, it is possible to derive closed form solutions for the thermodynamic conjugate to the field of c^M as

$$\hat{\mu}^{Mv} = \frac{\partial \hat{G}}{\partial c^{Mv}}, \quad v = f, m, \quad (76)$$

and ϵ_{ij}^{irr} , as

$$\hat{\mu}_{ij}^{irr v} = \frac{\partial \hat{G}}{\partial \epsilon_{ij}^{irr v}} = \frac{\partial \hat{G}^{eig pure}}{\partial \epsilon_{ij}^{irr v}} + \frac{\partial \hat{G}^{eig mix}}{\partial \epsilon_{ij}^{irr v}}, \quad v = f, m, \quad (77)$$

where

$$\begin{aligned} \bar{\rho} \frac{\partial \hat{G}^{eig pure}}{\partial \epsilon_{ij}^{irr v}} &= \rho^v \mu_{ij}^{irr v} = \rho^v b^{4v} \epsilon_{ij}^{irr v}, \\ \bar{\rho} \frac{\partial \hat{G}^{eig mix}}{\partial \epsilon_{ij}^{irr v}} &= -\sigma_{ij}^{Rv} = -\sigma_{ij}^{Rirr v} - \sigma_{ij}^{Rth v} - \sigma_{ij}^{*v}, \quad v = f, m. \end{aligned} \quad (78a - b)$$

II.4.3. Homogenization of the Strain Rates

The rate of the macroscopic total strain can be derived using (20), (21) and (29) as

$$\dot{E}_{ij} = \frac{1}{V_{meso}} \int_{\partial V_{meso}} \frac{1}{2} (\dot{u}_i n_j + \dot{u}_j n_i) ds = \dot{E}_{ij}^D = \dot{E}_{ij}^{De} + \dot{E}_{ij}^{Dth} + \dot{E}_{ij}^{Dirr}. \quad (79)$$

The strain rates \dot{E}_{ij}^{De} , \dot{E}_{ij}^{Dth} , and \dot{E}_{ij}^{Dirr} can be obtained using either of two methods: (1) By taking the time derivatives of (49) and (54); and/or (2) By applying the rate form of the macro-homogeneity condition (41), a specialized form of virtual power. Using the first method, the time derivative of (49) is given by

$$\begin{aligned} \dot{E}_{kl}^{De} &= \left(\overline{\dot{B}_{ijkl} a_{ijop}^1 B_{opmn}} + \overline{B_{ijkl} \Delta a_{ijop}^1 \dot{c}^M B_{opmn}} + \overline{B_{ijkl} a_{ijop}^1 \dot{B}_{opmn}} \right) \Sigma_{mn} \\ &+ \overline{B_{ijkl} a_{ijop}^1 B_{opmn}} \dot{\Sigma}_{mn}, \end{aligned} \quad (80)$$

where, using (57) and (58) from the first paper were used. Similarly, the time derivative of (54a) and (54b) are

$$\begin{aligned} \dot{E}_{kl}^{Dth} &= \overline{\dot{B}_{ijkl} [a_{ij}^2 \Delta T - a_{ij}^{2*} (T^* - T^o)]} + \overline{B_{ijkl} \Delta a_{ij}^2 \dot{c}^M \Delta T} + \overline{B_{ijkl} a_{ij}^2 \dot{T}}, \\ \dot{E}_{kl}^{Dirr} &= \overline{\dot{B}_{ijkl} (\epsilon_{ij}^{irr} - \epsilon_{ij}^{irr*})} + \overline{B_{ijkl} \Lambda_{ij} \dot{c}^M} + \overline{B_{ijkl} \dot{\epsilon}_{ij}^r}. \end{aligned} \quad (81)$$

For a two phase composite in which the material properties, state variables, and fluxes are assumed to be uniform within each phase, the decomposition of \dot{E}_{ij}^{irr} is similar to equation (61a) of the first paper:

$$\dot{E}_{ij}^{Dirr} = \dot{E}_{ij}^{Dt} + \dot{E}_{ij}^{Dr}, \quad (82)$$

where

$$\dot{E}_{ij}^{Dt} = c^f \dot{E}_{ij}^{Dtf} + c^m \dot{E}_{ij}^{Dtm}, \quad \dot{E}_{ij}^{Dr} = c^f \dot{E}_{ij}^{Drf} + c^m \dot{E}_{ij}^{Drm}, \quad (83a - b)$$

where

$$\dot{E}_{ij}^{Dtv} = \hat{\Lambda}_{ij}^v \dot{c}^{Mv}, \quad \dot{E}_{ij}^{Drv} = B_{kl ij}^v \dot{\epsilon}_{kl}^{rv}, \quad v = f, m, \quad (84a - b)$$

and the macroscopic transformation tensors, $\hat{\Lambda}_{ij}^v$, analogous to (62) of the first paper, are given by

$$\begin{aligned} \hat{\Lambda}_{ij}^f &= c^f \left[\frac{\partial B_{kl ij}^f}{\partial c^{Mf}} (\epsilon_{kl}^{irr f} - \epsilon_{kl}^{irr* f}) + B_{kl ij}^f \Lambda_{kl}^f \right] + c^m \frac{\partial B_{kl ij}^m}{\partial c^{Mf}} (\epsilon_{kl}^{irr m} - \epsilon_{kl}^{irr* m}), \\ \hat{\Lambda}_{ij}^m &= c^f \frac{\partial B_{kl ij}^f}{\partial c^{Mm}} (\epsilon_{kl}^{irr f} - \epsilon_{kl}^{irr* f}) + c^m \left[\frac{\partial B_{kl ij}^m}{\partial c^{Mm}} (\epsilon_{kl}^{irr m} - \epsilon_{kl}^{irr* m}) + B_{kl ij}^m \Lambda_{kl}^m \right]. \end{aligned} \quad (85a - b)$$

Alternative equations for the macroscopic strain rates can be obtained using the rate form of the macrohomogeneity condition. Toward this end, the kinematically admissible strains (38a-b) are converted to the kinematically admissible rates

$$\begin{aligned} \dot{\epsilon}_{ij}^{th} &= \dot{a}_{ijkl}^1 \sigma_{kl}^{Rth} + a_{ijkl}^1 \dot{\sigma}_{kl}^{Rth} + \dot{a}_{ij}^2 \Delta T + a_{ij}^2 \dot{T} \\ \dot{\epsilon}_{ij}^{irr} &= \dot{a}_{ijkl}^1 \sigma_{kl}^{Rirr} + a_{ijkl}^1 \dot{\sigma}_{kl}^{Rirr} + \dot{\epsilon}_{ij}^{irr}. \end{aligned} \quad (86a - b)$$

The total deformation strain rate can now be written as the sum of three kinematically admissible strain rates:

$$\dot{\epsilon}_{ij}^D = \dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^E + \dot{\epsilon}_{ij}^{th} + \dot{\epsilon}_{ij}^{irr}, \quad (87)$$

where the strain rate $\dot{\epsilon}_{ij}^E$ is associated by Hooke's law with the stress σ_{ij}^E as

$$\dot{\epsilon}_{ij}^E = \dot{a}_{ijkl}^1 \sigma_{kl}^E + a_{ijkl}^1 \dot{\sigma}_{kl}^E = \Delta a_{ijkl}^1 \dot{c}^M B_{kl mn} \Sigma_{mn} + a_{ijkl}^1 \dot{B}_{kl mn} \Sigma_{mn} + a_{ijkl}^1 B_{kl mn} \dot{\Sigma}_{mn}. \quad (88)$$

Application of the rate form of the macrohomogeneity condition,

$$\begin{aligned} \overline{\sigma_{ij}^\bullet \dot{\epsilon}_{ij}^E} &= \Sigma_{ij}^\bullet \dot{E}_{ij}^{De}, & \overline{\sigma_{ij}^\bullet \dot{\epsilon}_{ij}^{th}} &= \Sigma_{ij}^\bullet \dot{E}_{ij}^{Dth}, & \overline{\sigma_{ij}^\bullet \dot{\epsilon}_{ij}^{irr}} &= \Sigma_{ij}^\bullet \dot{E}_{ij}^{Dirr}, \\ \overline{\sigma_{ij}^\bullet \dot{\epsilon}_{ij}^D} &= \overline{\sigma_{ij}^\bullet \dot{\epsilon}_{ij}} = \Sigma_{ij}^\bullet \dot{E}_{ij}^D, \end{aligned} \quad (89a - d)$$

results in

$$\dot{E}_{kl}^{De} = \left(\overline{B_{ijkl} \Delta a_{ijop}^1 \dot{c}^M B_{opmn}} + \overline{B_{ijkl} a_{ijop}^1 \dot{B}_{opmn}} \right) \Sigma_{mn} + \overline{B_{ijkl} a_{ijop}^1 B_{opmn}} \dot{\Sigma}_{mn}, \quad (90)$$

$$\dot{E}_{kl}^{Dth} = \left(\overline{B_{ijkl} \Delta a_{ijop}^1 \sigma_{op}^{Rth}} + \overline{B_{ijkl} \Delta a_{ij}^2 \Delta T} \right) \dot{c}^M + \overline{B_{ijkl} a_{ij}^2 \dot{T}}, \quad (91)$$

and

$$\dot{E}_{kl}^{Dt} = \overline{\left(B_{ijkl} \Lambda_{ij} + B_{ijkl} \Delta a_{ijop}^1 \sigma_{op}^{Rirr} \right) \dot{c}^M}, \quad \dot{E}_{kl}^{Dr} = \overline{B_{ijkl} \dot{\epsilon}_{ij}^r}. \quad (92a - b)$$

Comparison of (90)-(92) with (80) and (81) yields the relationships

$$0 = \overline{\dot{B}_{ijkl} a_{ijop}^1 B_{opmn}}, \quad (93)$$

$$0 = \overline{\dot{B}_{ijkl} (\epsilon_{ij}^{irr} - \epsilon_{ij}^{irr*})} - \overline{B_{qrkl} \Delta a_{qrst}^1 \dot{c}^M \sigma_{st}^{Rirr}}, \quad (94)$$

and

$$0 = \overline{\dot{B}_{ijkl} [a_{ij}^2 \Delta T - a_{ij}^{2*} (T^* - T^o)]} - \overline{B_{qrkl} \Delta a_{qrst}^1 \dot{c}^M \sigma_{st}^{Rth}}. \quad (95)$$

II.4.4. Homogenization of the Dissipation

The macroscopic dissipation, $T \dot{\eta}$, is equal to the volume average of the mesoscale dissipation:

$$T \dot{\eta} = T \bar{\eta} = \overline{\Pi \dot{c}^M + \sigma_{ij}^{eff} \dot{\epsilon}_{ij}^r} = \overline{\sigma_{ij} \dot{\epsilon}_{ij}^{irr} - \rho \mu^M \dot{c}^M - \rho \mu_{ij}^{irr} \dot{\epsilon}_{ij}^{irr}} \geq 0. \quad (96)$$

The macroscopic irreversible power, $\Sigma_{ij} \dot{E}_{ij}^{Dirr}$, can be introduced into (96) by using the stress localization tensors and applying (35) and (38b) to obtain

$$\begin{aligned} T \dot{\eta} &= \Sigma_{ij} \left[\dot{E}_{ij}^{Dirr} - \overline{\dot{B}_{kl ij} (\epsilon_{kl}^{irr} - \epsilon_{kl}^{irr*})} \right] + \overline{(\sigma_{ij}^{Rirr} + \sigma_{ij}^{Rth} + \sigma_{ij}^*) \dot{\epsilon}_{ij}^{irr}} - \overline{\rho \mu^M \dot{c}^M} - \overline{\rho \mu_{ij}^{irr} \dot{\epsilon}_{ij}^{irr}} \\ &= \Sigma_{ij} \left[\dot{E}_{ij}^{Dirr} - \overline{\dot{B}_{kl ij} (\epsilon_{kl}^{irr} - \epsilon_{kl}^{irr*})} \right] + \overline{[(\sigma_{ij}^{Rirr} + \sigma_{ij}^{Rth} + \sigma_{ij}^* - \rho \mu_{ij}^{irr}) \Lambda_{ij} - \rho \mu^M] \dot{c}^M} \\ &\quad + \overline{(\sigma_{ij}^{Rirr} + \sigma_{ij}^{Rth} + \sigma_{ij}^* - \rho \mu_{ij}^{irr}) \dot{\epsilon}_{ij}^r} \geq 0. \end{aligned} \quad (97)$$

For a two phase composite in which the generalized forces and fluxes are uniform within each phase, (96) can be written as

$$T \dot{\eta} = T c^f \dot{\eta}^f + T c^m \dot{c}^m = c^f (\Pi^f \dot{c}^{Mf} + \sigma_{ij}^{eff f} \dot{\epsilon}_{ij}^{rf}) + c^m (\Pi^m \dot{c}^{Mm} + \sigma_{ij}^{eff m} \dot{\epsilon}_{ij}^{rm}) \geq 0. \quad (98)$$

Also, in this case (97) can be written as

$$T \dot{\eta} = \Sigma_{ij} \left[\dot{E}_{ij}^{Dirr} - \overline{\dot{B}_{kl ij} (\epsilon_{kl}^{irr} - \epsilon_{kl}^{irr*})} \right] - \bar{\rho} \sum_{v=f}^m \left(\frac{\partial \hat{G}}{\partial c^{Mv}} + \frac{\partial \hat{G}}{\partial \epsilon_{ij}^{irr v}} \Lambda_{ij}^v \right) \dot{c}^{Mv} - \bar{\rho} \sum_{v=f}^m \frac{\partial \hat{G}}{\partial \epsilon_{ij}^{irr v}} \dot{\epsilon}_{ij}^{rv} \geq 0. \quad (99)$$

Equations (82)-(85) can be used to rewrite (99) as

$$T \dot{\eta} = \sum_{v=f}^m \left[\left(\Sigma_{ij} c^v B_{kl ij}^v \Lambda_{kl}^v - \bar{\rho} \frac{\partial \hat{G}}{\partial \epsilon_{ij}^{irr v}} \Lambda_{ij}^v \right) - \bar{\rho} \frac{\partial \hat{G}}{\partial c^{Mv}} \right] \dot{c}^{Mv} + \sum_{v=f}^m \left(\Sigma_{ij} c^v B_{kl ij}^v - \bar{\rho} \frac{\partial \hat{G}}{\partial \epsilon_{kl}^{irr v}} \right) \dot{\epsilon}_{kl}^{rv} \geq 0. \quad (100)$$

Inspection of (100) and (98) results in

$$c^v \Pi^v = \Sigma_{ij} c^v B_{kl}^v \Lambda_{kl}^v - \bar{\rho} \frac{\partial \hat{G}}{\partial c^{Mv}} - \bar{\rho} \frac{\partial \hat{G}}{\partial \epsilon_{ij}^{irr v}} \Lambda_{ij}^v, \quad v = f, m, \quad (101)$$

and

$$\begin{aligned} \sigma_{ij}^{eff v} &= \Sigma_{kl} B_{ijkl}^v - (c^v)^{-1} \bar{\rho} \frac{\partial \hat{G}}{\partial \epsilon_{ij}^{irr v}} \\ &= \Sigma_{kl} B_{ijkl}^v + (c^v)^{-1} (\sigma_{ij}^{Rirr v} + \sigma_{ij}^{Rth v} + \sigma_{ij}^{*v} - \rho^v b^{4v} \epsilon_{ij}^{irr v}), \quad v = f, m. \end{aligned} \quad (102)$$

III. RESULTS AND DISCUSSION

III.1. Effective Transformation Temperatures

The composite will begin the transformation when the threshold criterion

$$\begin{aligned} \pm (2Y^{tv})^{\frac{1}{2}} &= \sigma_{ij}^{eff v} \Lambda_{ij}^v + \frac{1}{2} \Delta a_{ijkl}^{1v} \sigma_{ij}^v \sigma_{kl}^v + \Delta a_{ij}^{2v} \sigma_{ij}^v (T - T^{ov}) \\ &- \rho^v \Delta a^{3v} \left[(T - T^{ov}) - T \ln \left(\frac{T}{T^{ov}} \right) \right] + \rho^v \Delta a^{4v} T - \rho^v \Delta a^{5v} - \rho^v b^{7v} c^{Mv}, \quad v = f, m, \end{aligned} \quad (103)$$

is satisfied in phase v . When loading thermoelastically from the manufactured state, $\sigma_{ij}^{Rirr v} = 0$, and the composite stress-free ($\Sigma_{ij} = 0$) martensite start temperature, \hat{M}^s , can be obtained by substituting (36b) and (43) into (103),

$$\begin{aligned} (2Y^{tv})^{\frac{1}{2}} &= H^v \left[\frac{3}{2} (b_{ij}^{lv} \Delta T + \sigma_{ij}^{l*v} - \epsilon_{ij}^{lirr *v}) (b_{ij}^{lv} \Delta T + \sigma_{ij}^{l*v} - \epsilon_{ij}^{lirr *v}) \right]^{\frac{1}{2}} + \frac{1}{3} I^v (b_{pp}^v \Delta T + \sigma_{pp}^{*v}) \\ &+ \Delta a_{ij}^{2v} (b_{ij}^v \Delta T + \sigma_{ij}^{*v}) \Delta T + \frac{1}{2} \Delta a_{ijkl}^{1v} (b_{ij}^{lv} \Delta T + \sigma_{ij}^{l*v}) (b_{kl}^{lv} \Delta T + \sigma_{kl}^{l*v}) \\ &- \rho^v \Delta a^{3v} \left[\Delta T - T \ln \left(\frac{T}{T^o} \right) \right] + \rho^v \Delta a^{4v} T - \rho^v \Delta a^{5v}, \quad v = f, m, \end{aligned} \quad (104)$$

where $b_{ij}^{lv} = b_{ij}^v - \frac{1}{3} b_{pp}^v \delta_{ij}$. Note that the material must be loading in the forward sense:

$$\begin{aligned} &\frac{\partial \Pi^{forward v}}{\partial \sigma_{ij}^v} \dot{\sigma}_{ij} + \frac{\partial \Pi^{forward v}}{\partial T} \dot{T} = \\ &\left\{ \left[H^v \left[\frac{3}{2} (b_{mn}^{lv} \Delta T + \sigma_{mn}^{l*v} - \epsilon_{mn}^{lirr *v}) (b_{mn}^{lv} \Delta T + \sigma_{mn}^{l*v} - \epsilon_{mn}^{lirr *v}) \right]^{-\frac{1}{2}} (b_{ij}^{lv} \Delta T + \sigma_{ij}^{l*v} - \epsilon_{ij}^{lirr *v}) \right. \right. \\ &+ \frac{1}{3} I^v \delta_{ij} + \Delta a_{ij}^{2v} \Delta T + \left. \frac{1}{2} \Delta a_{ijkl}^{1v} (b_{kl}^{lv} \Delta T + \sigma_{kl}^{l*v}) \right] b_{ij}^v \\ &+ \Delta a_{ij}^{2v} (b_{ij}^v \Delta T + \sigma_{ij}^{*v}) + \rho^v \left[\Delta a^{3v} \ln \left(\frac{T}{T^o} \right) + \Delta a^{4v} \right] \left. \right\} \dot{T} \geq 0, \quad v = f, m. \end{aligned} \quad (105)$$

Similarly, when loading thermoelastically from the manufactured state, $\sigma_{ij}^{Rirr v} = 0$, and the composite stress-free ($\Sigma_{ij} = 0$) austenite start temperature, \hat{A}^s , can be obtained by substituting (36b) and (43) into (103),

$$\begin{aligned}
-(2Y^{tv})^{\frac{1}{2}} &= H \left[\left(\frac{2}{3} \epsilon_{mn}^{'irr *v} \epsilon_{mn}^{'irr *v} \right) \right]^{-\frac{1}{2}} (b_{ij}^{'v} \Delta T + \sigma_{ij}^{'*v} - \epsilon_{ij}^{'irr *v}) \epsilon_{ij}^{'irr *v} + \frac{1}{3} I^v (b_{pp}^{vM} \Delta T + \sigma_{pp}^{*v}) \\
&+ \Delta a_{ij}^{2v} (b_{ij}^{'v} \Delta T + \sigma_{ij}^{*v}) \Delta T + \frac{1}{2} \Delta a_{ijkl}^{1v} (b_{ij}^{'v} \Delta T + \sigma_{ij}^{*v}) (b_{kl}^{'v} \Delta T + \sigma_{kl}^{*v}) \\
&- \rho^v \Delta a^{3v} \left[\Delta T - T \ln \left(\frac{T}{T^0} \right) \right] + \rho^v \Delta a^{4v} T - \rho^v \Delta a^{5v}, \quad v = f, m,
\end{aligned} \tag{106}$$

and the material is loading in the reverse (negative) sense:

$$\begin{aligned}
\frac{\partial \Pi^{reverse v}}{\partial \sigma_{ij}^v} \dot{\sigma}_{ij} + \frac{\partial \Pi^{reverse v}}{\partial T} \dot{T} &= \\
\left\{ \left[H \left[\left(\frac{2}{3} \epsilon_{mn}^{'irr *v} \epsilon_{mn}^{'irr *v} \right) \right]^{-\frac{1}{2}} \epsilon_{ij}^{'irr *v} + \frac{1}{3} I^v \delta_{ij} + \Delta a_{ij}^{2v} \Delta T + \frac{1}{2} \Delta a_{ijkl}^{1v} (b_{kl}^{'v} \Delta T^* + \sigma_{kl}^{*v}) \right] b_{ij}^v \right. & \tag{107} \\
\left. + \Delta a_{ij}^{2v} (b_{ij}^{'v} \Delta T + \sigma_{ij}^{*v}) + \rho^v \left[\Delta a^{3v} \ln \left(\frac{T}{T^{ov}} \right) + \Delta a^{4v} \right] \right\} \dot{T} &\leq 0, \quad v = f, m.
\end{aligned}$$

III.2. The Mori-Tanaka Micromechanics Method

For a two phase composite in which the material properties, state variables, and fluxes are assumed to be uniform within each phase, the mesomechanics is completed by the specification of B_{ijkl}^f and B_{ijkl}^m . ESHELBY'S [1957] method, sometimes called the ‘‘equivalent inclusion method’’, is a means of obtaining the localization tensors of dilutely reinforced composite materials. The term ‘‘dilute’’ means that the reinforcement phase is of a sufficiently small volume fraction that the units of reinforcement, inhomogeneities such as particles, whiskers or fibers, do not interact. The MORI-TANAKA [1973] method is an approximate extension of Eshelby's method to the case of non-dilute volume fractions. Among the numerous references on the Mori-Tanaka method, we mention WENG [1984], MURA [1982], BENVENISTE [1987] and LAGOUDAS et al [1991]. Both Eshelby's method and the Mori-Tanaka method predict that the stress and strain fields are uniform within ellipsoidal inhomogeneities, but not in the matrix. However, the extension of the Mori-Tanaka method to the case with matrix eigenstrains assumes that the eigenstrains are uniform in the matrix. Thus, the stress localization tensors given by the Mori-Tanaka method are only approximate. A simple form of the fiber elastic stress concentration tensor is given by LAGOUDAS et al [1991]

$$B_{ijkl}^f = [I_{ijkl} + c^m (a_{ijmn}^{1m})^{-1} (I_{mnop} - S_{mnop}) (a_{opkl}^{1f} - a_{opkl}^{1m})]^{-1}. \tag{108}$$

The matrix localization tensor B_{ijkl}^m can be obtained from (32b). The solution of the mechanical problem requires the evaluation of the Eshelby tensor, S_{ijkl} , which is a function of the reinforcement shape and the matrix elastic compliance, a_{ijkl}^{1m} . For the special case of ellipsoidal inhomogeneities, closed form solutions are available for S_{ijkl} , which is a function of the inhomogeneity shape and the matrix elastic stiffness (MURA [1982]).

The incremental approach to inelastic mesomechanics can be formulated in either of two ways. First, as in the present paper, in terms of an elastic localization tensor, in which case the stresses are given by

$$\sigma_{ij}^v = B_{ijkl}^v \Sigma_{kl} + b_{ij}^v \Delta T + \mathcal{P}_{ijkl}^v \left[\left(\epsilon_{kl}^{irr f} - \epsilon_{kl}^{irr f*} \right) - \left(\epsilon_{kl}^{irr m} - \epsilon_{kl}^{irr m*} \right) \right], \quad v = f, m. \quad (109)$$

If the matrix elastic stiffness is constant, then it is not necessary to recalculate S_{ijkl} even if the matrix deforms inelastically. Alternatively, the mesomechanics problem could be formulated in terms of the tangent values of B_{ijkl}^v and b_{ij}^v , in which case

$$\dot{\sigma}_{ij}^v = B_{ijkl}^v \dot{\Sigma}_{kl} + b_{ij}^v \dot{T}, \quad v = f, m. \quad (110)$$

The tangent stress localization tensor is given by (108), but a_{ijkl}^{1v} in equation (108) is now the tangent compliance rather than the elastic compliance. The tangent compliance and tangent thermal expansion vector for the transformation response is given by (108a-b) in the first paper. Because the tangent stiffness varies during inelastic deformation of hardening materials, it is necessary to recalculate S_{ijkl} during each load step. Moreover, the Eshelby tensor must be evaluated numerically because inelastic deformation induces anisotropy in the tangent stiffness matrix of hardening materials (GAVAZZI and LAGOUDAS [1991]).

In the Mori-Tanaka method, the stress and strain are assumed to be uniform within the fiber, and the method does not use an actual solution to the fiber interaction problem. As pointed out by CHRISTENSEN [1990], “there has been no solution to the basic particle to particle interaction problem in the non-dilute case. In simplest terms the Mori-Tanaka method is an estimate of the solution form guided only by the requirement of the dilute solution at one end of the concentration scale and at the opposite end of the scale by the requirement that as $c_2 \rightarrow 1$, the effective property identify with that of the inclusion phase.” For the case of longitudinal loading of composites with stiff fibers reinforcing a compliant matrix, simple micromechanics methods are known to give accurate results because the stress state in the fiber is approximately uniaxial (longitudinal) and spatially homogeneous. However, for the case of transverse loading of “conventional” inelastic composites, i.e. composites consisting of an elastic fiber and an inelastic matrix, simple averaging methods are known to be inaccurate because of the complicated stress distribution in the matrix (LAGOUDAS et al [1991]). For the case of a SMA fiber in an elastomeric matrix, it has recently (LAGOUDAS et al [1993]) been demonstrated that the Mori-Tanaka results are in excellent agreement with the results of a finite element solution of the mesoscopic boundary value problem formulated over a representative volume element of a periodic composite. The simple solution is accurate in this case because transverse loading results in a relatively uniform spatial distribution of von Mises effective stress and strain within the transforming fiber.

III.3. Representative Thermomechanical Response

The constitutive response of the NiTi SMA fiber is given in the first paper. It is assumed that the matrix is an elastomer, which is approximated as an isotropic, linear elastic non-SMA material. In this case, the rate of the fiber elastic stress localization tensor is

$$\dot{B}_{ijkl}^f = \frac{\partial B_{ijkl}^f}{\partial c^{Mf}} \dot{c}^{Mf} = \left[-B_{ijrs}^f c^m (a_{rsmn}^{1m})^{-1} (I_{mnop} - S_{mnop}) \Delta a_{opuv}^{1f} B_{uvkl}^f \right] \dot{c}^{Mf}. \quad (111)$$

\dot{B}_{ijkl}^m can be derived using the rate form of (32b),

$$0 = c^f \dot{B}_{ijkl}^f + c^m \dot{B}_{ijkl}^m = c^f \frac{\partial B_{ijkl}^f}{\partial c^{Mf}} \dot{c}^{Mf} + c^m \frac{\partial B_{ijkl}^m}{\partial c^{Mf}} \dot{c}^{Mf}, \quad (112)$$

from which it follows that

$$\frac{\partial B_{ijkl}^m}{\partial c^{Mf}} = -\frac{c^f}{c^m} \frac{\partial B_{ijkl}^f}{\partial c^{Mf}}. \quad (113)$$

The rate of the irreversible eigenstress is

$$\dot{\sigma}_{ij}^{Rirr v} = \frac{\partial \mathcal{P}_{ijmn}^v}{\partial c^{Mf}} \dot{c}^{Mf} (\epsilon_{mn}^{irr f} - \epsilon_{mn}^{irr * m}) + \mathcal{P}_{ijmn}^v (\Lambda_{mn}^f \dot{c}^{Mf} + \dot{\epsilon}_{mn}^{rf}), \quad v = f, m, \quad (114)$$

where

$$\begin{aligned} \frac{\partial \mathcal{P}_{ijmn}^v}{\partial c^{Mf}} &= \left[-\frac{\partial B_{ijfg}^v}{\partial c^{Mf}} + (I_{ijkl} - B_{ijkl}^v)(a_{klst}^{1m} - a_{klst}^{1f})^{-1} \Delta a_{stfg}^{1f} \right] (a_{fgmn}^{1m} - a_{fgmn}^{1f})^{-1} \\ &= \left(-\frac{\partial B_{ijfg}^v}{\partial c^{Mf}} \mathcal{P}_{ijst}^v \Delta a_{stfg}^{1f} \right) (a_{fgmn}^{1m} - a_{fgmn}^{1f})^{-1}, \quad v = f, m. \end{aligned} \quad (115)$$

The rate of the thermal eigenstress is

$$\dot{\sigma}_{ij}^{Rth v} = \dot{b}_{ij}^v \Delta T + b_{ij}^v \dot{T} = \frac{\partial b_{ij}^v}{\partial c^{Mf}} \dot{c}^{Mf} \Delta T + b_{ij}^v \dot{T}, \quad v = f, m, \quad (116)$$

where

$$\frac{\partial b_{ij}^v}{\partial c^{Mf}} = \left(-\frac{\partial B_{ijrs}^v}{\partial c^{Mf}} + \mathcal{P}_{ijpq}^v \Delta a_{pqrs}^{1f} \right) (a_{rsmn}^{1m} - a_{rsmn}^{1f})^{-1} (a_{mn}^{2f} - a_{mn}^{2m}) + \mathcal{P}_{ijkl}^v \Delta a_{kl}^{2f}, \quad v = f, m. \quad (117)$$

Unless otherwise stated, the matrix elastic modulus E^m , Poisson ratio ν^m , and thermal expansion coefficient α^m are

$$E^m = 2.0 \times 10^3 \text{ MPa} \quad \nu^m = 0.45 \quad \alpha^m = 75.0 \times 10^{-6} / \text{K}. \quad (118)$$

In the following results, the fiber volume fraction is assumed to be 30%, i.e. $c^f = 0.30$. In addition, it is assumed that the reference temperature corresponds to the manufacturing temperature, i.e. $T^\circ = T^*$, so (43) is applicable.

The temperature (equation (67)) at which the composite will become thermally unstable is approximately 470 K. At this temperature, the stresses required to initiate the martensitic transformation will exceed the dislocation yield strength. As a practical matter, the present theory is not valid at such high temperatures because it does not include the effects of plastic deformation due to dislocation movement.

The composite pseudoelastic response due to transformation, Figures 3a and 3b, is determined for two temperatures, $T^* = 323 K$ and $T^* = 343 K$. In each case, the manufactured state is trivial:

$$0 \equiv c^{M*f} = \epsilon_{ij}^{irr*f} = \sigma_{ij}^{*f} = \sigma_{ij}^{*m}. \quad (119)$$

Two loading conditions are considered: (1) Isothermal, uniaxial longitudinal stress $0 = \Sigma_{33}$, $ij \neq 33$, for which Σ_{33} vs E_{33}^D and Σ_{33} vs E_{22}^D are plotted; and (2) Isothermal, uniaxial transverse stress $0 = \Sigma_{22}$, $ij \neq 22$, for which Σ_{22} vs E_{22}^D and Σ_{22} vs E_{33}^D are plotted.

Figure 4a corresponds to the same loading conditions of Figures 3a and 3b, but at a lower temperature, $T = 280K$. At this temperature, the composite does not undergo the reverse transformation during unloading. However, if the composite is heated, the reverse transformation occurs, resulting in the recovery of the transformation strain. This one-way SME is indicated in Figures 4a and 4b for the manufactured state given by (119). The macroscopic stress is zero in Figure 4b, i.e. $\Sigma_{ij} = 0$. The longitudinally loaded composite begins the reverse transformation soon after heating, whereas the transversely loaded composite requires substantial heating to initiate the reverse transformation. This difference between the two cases is due to the irreversible eigenstresses, σ_{ij}^{Rirr} , that are generated during the forward transformation.

During phase transformation, proportional macroscopic stressing does not result in proportional mesoscopic stressing. This phenomena is shown in Figure 5, which corresponds to monotonic transverse stressing ($\Sigma_{ij} = 0$, $ij \neq 22$) of the composite with the trivial manufactured state, (119), and $T^* = 323 K$. Throughout the phase transformation, the reorientation criteria (69) of the first paper are satisfied. Furthermore, because the fiber stressing is non-proportional, the kinematic constraint (68) of the first paper allows for simultaneous fiber transformation and reorientation. At the completion of the forward transformation, $\epsilon_{22}^{irr*f} = 0.0500$, $\epsilon_{22}^{t*f} = 0.0489$, and $\epsilon_{22}^{r*f} = 0.0011$. The amount of reorientation strain is small because the degree of non-proportionality in the fiber is small. In contrast, the strong non-proportional loading of Figure 8a of the first paper leads to significant simultaneous transformation and reorientation, as indicated in Figure 8b of the first paper.

Figure 6 illustrates the composite's ability to undergo the TWSME even though the SMA fiber can only undergo the one-way SME. The manufactured state for Figure 6 corresponds to the case of completely oriented martensite:

$$\begin{aligned} T^* = 280 K, \quad c^{M*f} = 1.0, \quad \epsilon_{33}^{irr*f} = -2\epsilon_{22}^{irr*f} = -2\epsilon_{11}^{irr*f} = 0.050, \\ 0 = \epsilon_{23}^{irr*f} = \epsilon_{13}^{irr*f} = \epsilon_{12}^{irr*f}, \quad 0 = \sigma_{ij}^{*f} = \sigma_{ij}^{*m}. \end{aligned} \quad (120)$$

The thermal loading from the manufactured state consists of heating from $T^* = 280 K$ to complete the reverse transformation, during which c^{M*f} decreases from 1.0 to 0.0, followed by cooling to complete the forward transformation, during which c^{M*f} increases from 0.0 to 1.0. The macroscopic stress is zero, i.e. $\Sigma_{ij} = 0$. The two curves correspond to two different values for the matrix elastic modulus, $E^m = 2$ GPa and $E^m = 4$ GPa.

There are four significant features in Figure 6. First, both materials begin the reverse transformation at approximately the same temperature, i.e. \hat{A}^s is similar for both materials because the

purely thermal term, $\rho^f \Delta a^{4f} T - \rho^f \Delta a^{5f}$, is the dominant part of (106). Second, the stiff matrix composite completes both the reverse and the forward transformations over a much greater temperature range. This phenomenon can be explained by considering the flow rule, equation (76) from the first paper, for the fiber:

$$\dot{c}^{Mf} = (\mathcal{D}^{1f})^{-1} \left(\frac{\partial \Pi^f}{\partial \sigma_{ij}^f} \dot{\sigma}_{ij}^f + \frac{\partial \Pi^f}{\partial T} \dot{T} \right). \quad (121)$$

c^{M*f} and ϵ_{ij}^{*f} are the same for both curves in Figure 6. Therefore, the hardening term \mathcal{D}^{1f} , equation (77) in the first paper, is the same in each case. The numerator in (121) is explicitly written in (107), in which b_{ij}^f is the only term that differs between the two cases. In (107), the positive product $\epsilon_{ij}^{'irr*} b_{ij}^f$, which is larger for the stiff matrix composite, reduces the effect of the large negative term Δa^{4f} , thereby slowing the reverse transformation rate, \dot{c}^{Mf} , during heating. However, for a given \dot{c}^{Mf} , the transformation strain rate

$$\dot{\epsilon}_{ij}^{'irr f} = \Lambda_{ij}^f \dot{c}^{Mf} = H \left[\left(\frac{2}{3} \epsilon_{mn}^{'irr f} \epsilon_{mn}^{'irr f} \right) \right]^{-\frac{1}{2}} \epsilon_{ij}^{'irr f} \quad (122)$$

is the same in each case because Λ_{ij}^f is independent of stress during the reverse transformation. The third outstanding feature of Figure 6 is that the composite strain E_{33}^D is greater for the compliant matrix composite. This is because $E_{33}^D = E_{33}^{De} + E_{33}^{Dth} + E_{33}^{Dirr}$, $E_{ij}^{De} = 0$, and E_{33}^{Dirr} , which is much greater than E_{33}^{Dth} , is given by

$$E_{33}^{Dirr} = \overline{B_{ij33}(\epsilon_{ij}^{'irr} - \epsilon_{ij}^{'irr*})} = c^f B_{ij33}^f (\epsilon_{ij}^{'irr f} - \epsilon_{ij}^{'irr*f}), \quad (123)$$

and B_{ij33}^f is greater for the compliant matrix composite. The fourth outstanding feature of Figure 6 is that the compliant matrix composite does not complete a strain cycle during a complete cycle of T and c^{Mf} . This phenomenon can be explained by considering the fiber transformation strain rate during the forward transformation:

$$\dot{\epsilon}_{ij}^{'irr f} = \Lambda_{ij}^f \dot{c}^{Mf} = \frac{3}{2} H (\bar{\sigma}^{eff f})^{-1} \sigma_{ij}^{'eff f}, \quad (124)$$

where

$$\begin{aligned} \sigma_{ij}^{'eff f} &= \sigma_{ij}^{'Rirr f} + \sigma_{ij}^{'Rth f} - \rho^f b^{4f} \epsilon_{ij}^{'irr f} \\ &= \left[\mathcal{P}_{ijkl}^f (\epsilon_{kl}^{'irr f} - \epsilon_{kl}^{'irr*f}) \right]' + \left[\mathcal{P}_{ijkl}^f (a_{kl}^{2f} - a_{kl}^{2m}) \right]' \Delta T - \rho^f b^{4f} \epsilon_{ij}^{'irr f}. \end{aligned} \quad (125)$$

For the compliant matrix composite, the longitudinal component of each of the four terms in (125) are plotted in Figure 7. The units of time are arbitrary because the equations are rate-independent. Near the end of the forward transformation, $\sigma_{33}^{'eff f}$ becomes zero. In fact, $\sigma_{33}^{'Rth f}$ goes from positive to negative as the temperature cools below $T^* = 280 K$. Subsequent thermal cycles will result in E_{33}^D being completely recovered with respect to the E_{33}^D at the end of the first cycle.

In Figure 8, the manufactured state and deformational thermomechanical loading are identical to Figure 6, except that the effects of the processing induced longitudinal residual stresses σ_{33}^{*f} are included in Figure 8. As indicated in Figure 8 and equations (104) and (105), a compressive σ_{33}^{*f} lowers \hat{A}^s and raises \hat{M}^s , whereas a tensile σ_{33}^{*f} raises \hat{A}^s and lowers \hat{M}^s . In this case, (125) becomes

$$\begin{aligned} \sigma_{ij}^{ieff} &= \sigma_{ij}^{iRirr} + \sigma_{ij}^{iRth} + \sigma_{ij}^{i*f} - \rho^f b^{4f} \epsilon_{ij}^{iirr} \\ &= \left[\mathcal{P}_{ijkl}^f (\epsilon_{kl}^{iirr} - \epsilon_{kl}^{iirr *f}) \right]' + \left[\mathcal{P}_{ijkl}^f (a_{kl}^{2f} - a_{kl}^{2m}) \right]' \Delta T + \sigma_{ij}^{i*f} - \rho^f b^{4f} \epsilon_{ij}^{iirr} , \end{aligned} \quad (126)$$

which helps to explain why E_{33}^D is fully recovered with tensile σ_{33}^{*f} , but not with compressive σ_{33}^{*f} .

Relative to the monolithic SMA material, the SMA composite allows for greater control of the transformation parameters. For example, alloying on the microscale of monolithic SMA materials is commonly used to control the threshold temperatures M^s , M^f , A^s , A^f , where the superscript “f” denotes “finish”, not fiber. However, as indicated in Figure 8, it is possible to use the mesoscale mechanical fields σ_{ij}^* to control the SMA composite threshold temperatures \hat{M}^s , \hat{M}^f , \hat{A}^s , \hat{A}^f , and the composite transformation strain, E_{ij}^{Dirr} . In addition, as indicated in Figure 6, the matrix elastic stiffness can be used to control E_{ij}^{Dirr} and the temperature ranges ($\hat{M}^s - \hat{M}^f$) and ($\hat{A}^s - \hat{A}^f$). This ability to engineer the mesoscale properties may allow for easy and affordable manufacturing of active, functionally gradient SMA structures.

IV. SUMMARY

As indicated in Figure 1, the phenomenological equations developed in the first paper were used in this second paper to mesomechanically derive the macroscopic free energy and dissipation of a SMA composite material. The mesomechanics includes the effects of processing-induced microstructural changes. For a two phase composite, the macroscopic state variables are $(\Sigma_{ij}, T, c^{Mf}, c^{Mm}, \epsilon_{ij}^{iirr}, \epsilon_{ij}^{iirr m})$. The Mori-Tanaka micromechanics method was used to obtain the fiber and matrix elastic stress localization tensors. It was demonstrated that a SMA fiber/elastomer matrix composite can undergo the TWSME even though the SMA fiber can only undergo the one-way SME. Furthermore, it was found that processing-induced residual stresses can be used to control the composite austenite start and martensite start temperatures, as well as the amount of composite strain recovered during a thermal cycle. Relative to stiff-matrix composites, compliant-matrix composites were shown to: (1) Complete the phase transformation over a narrower temperature range; (2) Exhibit greater transformation strain during the reverse transformation; and (3) Undergo an incomplete strain cycle during a complete cycle of T and fiber martensitic volume fraction, c^{Mf} . Finally, it was demonstrated that during transformation, macroscopic proportional stressing of the composite results in non-proportional fiber stressing, which in turn causes a small amount of martensitic reorientation to occur simultaneously with the transformation.

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Table 1. SMA fiber and elastomer matrix properties

NiTi material properties

$$D^A = 30.0 \times 10^3 \text{ MPa}$$

$$D^M = 13.0 \times 10^3 \text{ MPa}$$

$$\nu = 0.33$$

$$\alpha^A = 11.0 \times 10^{-6} / ^\circ C$$

$$\alpha^M = 6.6 \times 10^{-6} / ^\circ C$$

$$M^s = 23^\circ C$$

$$M^f = 5^\circ C$$

$$A^s = 29^\circ C$$

$$A^f = 51^\circ C$$

$$C^M = 11.3 \text{ MPa}/^\circ C$$

$$C^A = 4.5 \text{ MPa}/^\circ C$$

$$\Omega = - 0.91 \times 10^3 \text{ MPa}$$

Polymer material properties

$$D = 2.0 \times 10^3 \text{ MPa}$$

$$\nu = 0.45$$

$$\alpha = 75 \times 10^{-6} / ^\circ C$$

Material	$(\frac{3}{2}B'_{ij33}{}^{fea} B'_{ij33}{}^{fea})^{\frac{1}{2}}$	$(\frac{3}{2}B'_{ij33}{}^{fem} B'_{ij33}{}^{fem})^{\frac{1}{2}}$	$(\frac{3}{2}b'_{ij}{}^{fea} b'_{ij}{}^{fea})^{\frac{1}{2}}$	$(\frac{3}{2}b'_{ij}{}^{fem} b'_{ij}{}^{fem})^{\frac{1}{2}}$
elastomer	3.950	3.101	0.390	0.320
aluminum	0.455	0.208	0.187	0.077
tool steel	0.178	0.079	0.000	0.010
isotropic graphite	0.039	0.015	0.018	0.001

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Table 1. Material NiTi material properties

$$\begin{aligned}
E^A &= 30.0 \times 10^3 \text{ MPa} \\
E^M &= 13.0 \times 10^3 \text{ MPa} \\
\nu &= 0.33 \\
\alpha^A &= 11.0 \times 10^{-6} / ^\circ\text{C} \\
\alpha^M &= 6.6 \times 10^{-6} / ^\circ\text{C} \\
b^2 &= 7.519 \text{ MJ/kg} \\
b^4 &= 324.7 \times 10^{-4} \text{ MJ/kg} \\
\Delta a^3 &= -57.84 \times 10^{-6} \text{ MJ/(kg } ^\circ\text{K)} \\
\Delta a^4 &= -17.00 \times 10^{-3} \text{ MJ/kg} \\
\Delta a^5 &= 3.222 \times 10^{-4} \text{ MJ/(kg } ^\circ\text{K)} \\
H &= 0.050 \\
I &= 0.000 \\
Y^t &= 30.00 \text{ (MPa)}^2 \\
Y^r &= 5000 \text{ (MPa)}^2
\end{aligned}$$

Discussion. In the case of plasticity, it is well established (Mandel, 1971; Hill and Rice, 1973; Suquet, 1982) that the global plastic potential is equal to the volume average of the local plastic potential,

$$\hat{\phi}^p \{ \Sigma_{ij}, \sigma_{ij}^{Rp} \} = \overline{\phi^p} \quad (98)$$

such that the global plastic strain rate is given by

$$\dot{E}_{ij}^p = \frac{\partial \hat{\phi}^p}{\partial \Sigma_{ij}} = \frac{\overline{\partial \phi^p}}{\partial \sigma_{kl}} \frac{\partial \sigma_{kl}}{\partial \Sigma_{ij}} = \overline{B_{kl ij} \dot{\epsilon}_{kl}^p}. \quad (99)$$

Compare to the current case.

$$E_{kl} = E_{kl}^e + E_{kl}^{th} + E_{kl}^t + E_{kl}^r = \overline{a_{klmn}^1 B_{mnop}} \Sigma_{op} + \overline{B_{ijkl} a_{ij}^2} \Delta T + \overline{B_{ijkl} \epsilon_{ij}^t} + \overline{B_{ijkl} \epsilon_{ij}^r}. \quad (76)$$

The total strain rate is (IS THIS VALID FOR TRANSFORMATION STRAIN IN THE MATRIX ??)

$$\begin{aligned}
\dot{E}_{kl} &= \overline{a_{klmn}^1 B_{mnop}} \dot{\Sigma}_{op} + \overline{B_{ijkl} a_{ij}^2} \dot{T} + \overline{B_{ijkl} \dot{\epsilon}_{ij}^r} \\
&+ \overline{\left[\left(\frac{\partial a_{klmn}^1}{\partial c^M} B_{mnop} + a_{klmn}^1 \frac{\partial B_{mnop}}{\partial c^M} \right) \Sigma_{op} + \left(\frac{\partial B_{ijkl}}{\partial c^M} a_{ij}^2 + B_{ijkl} \frac{\partial a_{ij}^2}{\partial c^M} \right) \right.} \\
&\left. + \frac{\partial B_{ijkl}}{\partial c^M} (\epsilon_{ij}^t + \epsilon_{ij}^r) + B_{ijkl} \Lambda_{ij} \right] \dot{c}^M} \quad (1)
\end{aligned}$$

B_{ijkl} changes because it is a function of a_{ijkl}^1 , which is a function of c^M .

Because $a_{ijkl}^1 B_{ijmn} \Sigma_{mn}$ is K.A. and σ_{ij}^R is S.E., application of (58) leads to the simplified form

$$\begin{aligned} \overline{\rho G^e} = & -\frac{1}{2} A_{mnop}^1 \Sigma_{mn} \Sigma_{op} - A_{op}^2 \Sigma_{op} T - \overline{a_{ij}^2 B_{ijop} T^\circ \Sigma_{op}} \\ & - \frac{1}{2} \overline{a_{ijkl}^1 (\sigma_{ij}^{Rirr} + b_{ij} \Delta T^* + \sigma_{ij}^*) (\sigma_{kl}^{Rirr} + b_{kl} \Delta T^* + \sigma_{kl}^*)} - \overline{a_{ij}^2 (\sigma_{ij}^{Rirr} + b_{ij} \Delta T^* + \sigma_{ij}^*) \Delta T} \end{aligned} \quad (202)$$

where

$$A_{mnop}^1 = \overline{a_{ijkl}^1 B_{ijmn} B_{klon}}, \quad A_{op}^2 = \overline{a_{ij}^2 B_{ijop}}. \quad (203)$$

This next part is interesting, but maybe not really necessary

Equations (62) and (58) can be used to explicitly include the irreversible local eigenstrains in the global free energy, resulting in

$$\begin{aligned} \overline{\rho G^e} = & -\frac{1}{2} A_{mnop}^1 \Sigma_{mn} \Sigma_{op} - A_{op}^2 \Sigma_{op} T - \overline{a_{ij}^2 B_{ijop} T^\circ \Sigma_{op}} \\ & + \frac{1}{2} \overline{(\sigma_{ij}^{Rirr} + b_{ij} \Delta T^* + \sigma_{ij}^*) (\epsilon_{ij}^{irr} + a_{ij}^2 \Delta T)} - \overline{(\sigma_{ij}^{Rirr} + b_{ij} \Delta T^* + \sigma_{ij}^*) a_{ij}^2 \Delta T} \\ & - \frac{1}{2} \overline{a_{ijkl}^1 [(a_{ijmn}^1)^{-1} - (a_{ijmn}^{1*})^{-1}] (\sigma_{ij}^{Rirr} + b_{ij} \Delta T^* + \sigma_{ij}^*) [\epsilon_{mn}^{irr*} + a_{mn}^{2*} (T^* - T^\circ)]} \end{aligned} \quad (204)$$

But note that

$$\begin{aligned} \overline{b_{ij} a_{ij}^2 \Delta T \Delta T^*} = & \overline{\sigma_{ij}^{Rth} \epsilon_{ij}^{th}} = \overline{\sigma_{ij}^{Rth} [\zeta_{ij}^{th} - a_{ijkl}^1 \sigma_{kl}^{Rth} + a_{ij}^{2*} (T^* - T^\circ)]} \\ & - \overline{a_{ijkl}^1 b_{ij} b_{kl} (\Delta T^*)^2} + \overline{a_{ij}^{2*} b_{ij} (T^* - T^\circ) \Delta T^*}. \end{aligned} \quad (205)$$

$$\begin{aligned} \hat{\rho} \hat{G}^{eig\ mix} = & \frac{1}{2} \left[\mathcal{P}_{ijkl}^m \epsilon_{ij}^{irr\ m} + c^f (\mathcal{P}_{ijkl}^f \epsilon_{ij}^{irrf} - \mathcal{P}_{ijkl}^m \epsilon_{ij}^{irrm}) \right] (\epsilon_{kl}^{irrf} - \epsilon_{kl}^{irrm}) \\ & + \frac{1}{2} \left[b_{ij}^m \epsilon_{ij}^{irr\ m} + c^f (b_{ij}^f \epsilon_{ij}^{irrf} - b_{ij}^m \epsilon_{ij}^{irrm}) \right] \Delta T^* \\ & - \frac{1}{2} \left[\mathcal{P}_{ijkl}^m a_{ij}^{2m} + c^f (\mathcal{P}_{ijkl}^f a_{ij}^{2f} - \mathcal{P}_{ijkl}^m a_{ij}^{2m}) \right] (\epsilon_{kl}^{irrf} - \epsilon_{kl}^{irrm}) \Delta T \end{aligned} \quad (413)$$

Also

$$\begin{aligned} \hat{\rho} \hat{G}^{eig\ pure} = & c^f \rho^f \frac{1}{2} b^{7f} (c^{Mf})^2 + c^m \rho^m \frac{1}{2} b^{7m} (c^{Mm})^2 \\ & + c^f \rho^f \frac{1}{2} b^{4f} \epsilon_{ij}^{irrf} \epsilon_{ij}^{irrf} + c^m \rho^m \frac{1}{2} b^{4m} \epsilon_{ij}^{irrm} \epsilon_{ij}^{irrm}. \end{aligned} \quad (406)$$

The macroscopic thermoelastic deformation strain is

$$\begin{aligned} E_{ij}^{Dte} &= E_{ij}^{De} + E_{ij}^{Dth} = -\frac{1}{\bar{\rho}} \frac{\partial \bar{G}}{\partial \Sigma_{ij}} - \frac{\partial(\bar{\rho} \bar{G})}{\partial \Sigma_{ij}}, \\ E_{ij}^{De} &= A_{ijkl}^1 \Sigma_{kl}, \quad E_{ij}^{Dth} = A_{ij}^2 T - \overline{a_{op}^2 B_{opij} T^o} \end{aligned} \quad (800)$$

Noting that the stress localization tensor is a function of a_{ijkl}^1 , (306)-(309) can be rewritten as

$$\begin{aligned} \dot{E}_{kl}^{De} &= \overline{\left(\frac{\partial B_{ijkl}}{\partial a_{qrst}^1} a_{ijop}^1 B_{opmn} + B_{qrkl} B_{stmn} + B_{ijkl} a_{ijop}^1 \frac{\partial B_{opmn}}{\partial a_{qrst}^1} \right) \Delta a_{qrst}^1 \dot{c}^M \Sigma_{mn}} \\ &+ \overline{B_{ijkl} a_{ijop}^1 B_{opmn}} \dot{\Sigma}_{mn}, \quad \text{where} \quad \Delta a_{qrst}^1 = \frac{\partial a_{qrst}^1}{\partial c^M}, \end{aligned} \quad (704)$$

and

$$\dot{E}_{kl}^{Dt} = \overline{\left[\frac{\partial B_{ijkl}}{\partial a_{qrst}^1} \Delta a_{qrst}^1 (\epsilon_{ij}^{irr} - \epsilon_{ij}^{irr*}) + B_{ijkl} \Lambda_{ij} \right] \dot{c}^M}, \quad \dot{E}_{kl}^{Dr} = \overline{B_{ijkl} \dot{\epsilon}_{ij}^r}, \quad (705a - b)$$

and

$$\begin{aligned} \dot{E}_{kl}^{Dth} &= \overline{\left\{ \frac{\partial B_{ijkl}}{\partial a_{qrst}^1} \Delta a_{qrst}^1 [a_{ij}^2 \Delta T - a_{ij}^{2*} (T^* - T^o)] + B_{ijkl} \Delta a_{ij}^2 \Delta T \right\} \dot{c}^M} \\ &+ \overline{B_{ijkl} a_{ij}^2 \dot{T}}, \quad \text{where} \quad \frac{\partial a_{ij}^2}{\partial c^M} = \Delta a_{ij}^2. \end{aligned} \quad (706)$$

Note that Δa_{ijkl}^1 and Δa_{ij}^2 are defined in equations ?? and ?? of the first paper.

***** Note that Δa_{ijkl}^1 and Δa_{ij}^2 are defined in equations ?? and ?? of the first paper.

Now use virtual power:

$$\begin{aligned} \dot{\epsilon}_{ij}^{th} &= \dot{a}_{ijkl}^1 \sigma_{kl}^{Rth} + a_{ijkl}^1 \dot{\sigma}_{kl}^{Rth} + \dot{a}_{ij}^2 \Delta T + a_{ij}^2 \dot{T} = \Delta a_{ijkl}^1 \dot{c}^M \sigma_{kl}^{Rth} + a_{ijkl}^1 \dot{\sigma}_{kl}^{Rth} + \Delta a_{ij}^2 \dot{c}^M \Delta T + a_{ij}^2 \dot{T} \\ \dot{\epsilon}_{ij}^{irr} &= \dot{a}_{ijkl}^1 \sigma_{kl}^{Rirr} + a_{ijkl}^1 \dot{\sigma}_{kl}^{Rirr} + \dot{\epsilon}_{ij}^{irr}, \end{aligned} \quad (730a - b)$$

The total deformation strain rate can now be written as the sum of three kinematically admissible strain rates:

$$\dot{\epsilon}_{ij}^D = \dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^E + \dot{\epsilon}_{ij}^{th} + \dot{\epsilon}_{ij}^{irr}, \quad (209)$$

where the strain rate $\dot{\epsilon}_{ij}^E$ is associated by Hooke's law with the stress σ_{ij}^E as

$$\dot{\epsilon}_{ij}^E = \dot{a}_{ijkl}^1 \dot{\sigma}_{kl}^E \Delta a_{ijkl}^1 \sigma_{kl}^E a_{ijkl}^1 \dot{B}_{klmn} \Sigma_{mn} + a_{ijkl}^1 B_{klmn} \dot{\Sigma}_{mn} \quad (210)$$

Note that ϵ_{ij}^E is kinematically admissible because it is the total strain that results from an isothermal elastic input E_{ij}^{De} . Application of the macrohomogeneity condition results in

$$\begin{aligned}\overline{\sigma_{ij}^\bullet \dot{\epsilon}_{ij}^E} &= \Sigma_{ij}^\bullet \dot{E}_{ij}^{De}, & \overline{\sigma_{ij}^\bullet \dot{\epsilon}_{ij}^{th}} &= \Sigma_{ij}^\bullet \dot{E}_{ij}^{Dth}, & \overline{\sigma_{ij}^\bullet \dot{\epsilon}_{ij}^{irr}} &= \Sigma_{ij}^\bullet \dot{E}_{ij}^{Dirr}, \\ \overline{\sigma_{ij}^\bullet \dot{\epsilon}_{ij}^D} &= \overline{\sigma_{ij}^\bullet \dot{\epsilon}_{ij}} = \Sigma_{ij}^\bullet \dot{E}_{ij}^D,\end{aligned}\quad (211a - f)$$

$$\begin{aligned}\overline{\rho G^e} &= -\frac{1}{2} A_{mnop}^1 \Sigma_{mn} \Sigma_{op} - A_{op}^2 \Sigma_{op} T + \overline{a_{ij}^2 B_{ijop} T^o} \Sigma_{op} \\ &+ \frac{1}{2} (\overline{\sigma_{ij}^{Rirr} + \sigma_{ij}^{Rth} + \sigma_{ij}^*}) \{ (\overline{\epsilon_{ij}^{irr} - \epsilon_{ij}^{irr*}}) + [a_{ij}^2 \Delta T + a_{ij}^{2*} (T^* - T^o)] \} \\ &- \frac{1}{2} \overline{a_{ijkl}^1 (a_{ijmn}^{1*})^{-1} (\sigma_{kl}^{Rirr} + \sigma_{kl}^{Rth} + \sigma_{kl}^*) [\epsilon_{mn}^{irr*} + a_{mn}^{2*} (T^* - T^o)]}.\end{aligned}\quad (354)$$

Equation (204) can also be written as

$$\begin{aligned}\overline{\rho G^e} &= -\frac{1}{2} A_{mnop}^1 \Sigma_{mn} \Sigma_{op} - A_{op}^2 \Sigma_{op} T + \overline{a_{ij}^2 B_{ijop} T^o} \Sigma_{op} \\ &+ \frac{1}{2} \overline{(\sigma_{ij}^{Rirr} + \sigma_{ij}^{Rth} + \sigma_{ij}^*) (\epsilon_{ij}^{irr} - a_{ij}^2 \Delta T)} \\ &- \frac{1}{2} \overline{a_{ijkl}^1 [(a_{ijmn}^{1*})^{-1} - (a_{ijmn}^{1*})^{-1}] (\sigma_{kl}^{Rirr} + \sigma_{kl}^{Rth} + \sigma_{kl}^*) [\epsilon_{mn}^{irr*} + a_{mn}^{2*} (T^* - T^o)]}.\end{aligned}\quad (355)$$

Note that the last term in (204) vanishes if $a_{ijkl}^{1*} = a_{ijkl}^1$. Also,

$$\begin{aligned}\overline{\sigma_{ij}^{Rth} a_{ij}^2 \Delta T} &= \overline{\sigma_{ij}^{Rth} \epsilon_{ij}^{th}} = \overline{\sigma_{ij}^{Rth} [\tilde{\epsilon}_{ij}^{th} - a_{ijkl}^1 \sigma_{kl}^{Rth} + a_{ij}^{2*} (T^* - T^o)]} \\ &= -\overline{a_{ijkl}^1 \sigma_{ij}^{Rth} \sigma_{kl}^{Rth}} + \overline{\sigma_{ij}^{Rth} a_{ij}^{2*} (T^* - T^o)}.\end{aligned}\quad (356)$$

Equation (356) can be used in (355), resulting in

$$\begin{aligned}\overline{\rho G^e} &= -\frac{1}{2} A_{mnop}^1 \Sigma_{mn} \Sigma_{op} - A_{op}^2 \Sigma_{op} T + \overline{a_{ij}^2 B_{ijop} T^o} \Sigma_{op} \\ &+ \frac{1}{2} \overline{(\sigma_{ij}^{Rirr} + \sigma_{ij}^{Rth} + \sigma_{ij}^*) \epsilon_{ij}^{irr}} - \frac{1}{2} \overline{(\sigma_{ij}^{Rirr} + \sigma_{ij}^*) a_{ij}^2 \Delta T} \\ &+ \frac{1}{2} \overline{a_{ijkl}^1 \sigma_{ij}^{Rth} \sigma_{kl}^{Rth}} - \frac{1}{2} \overline{\sigma_{ij}^{Rth} a_{ij}^{2*} (T^* - T^o)} \\ &- \frac{1}{2} \overline{a_{ijkl}^1 [(a_{ijmn}^{1*})^{-1} - (a_{ijmn}^{1*})^{-1}] (\sigma_{kl}^{Rirr} + \sigma_{kl}^{Rth} + \sigma_{kl}^*) [\epsilon_{mn}^{irr*} + a_{mn}^{2*} (T^* - T^o)]}\end{aligned}\quad (357)$$

When (212) is applicable (DIMITRIS: SEE THE NOTE TO YOU ON PAGE 7. $a_{ij}^{2*} = a_{ij}^2$ or

$T^\circ = T^*$), (357) can be rewritten as

$$\begin{aligned}
\overline{\rho G^e} &= -\frac{1}{2}A_{mnop}^1 \Sigma_{mn} \Sigma_{op} - A_{op}^2 \Sigma_{op} T + \overline{a_{ij}^2 B_{ijop} T^\circ \Sigma_{op}} \\
&+ \frac{1}{2} \overline{(\sigma_{ij}^{Rirr} + b_{ij} \Delta T^* + \sigma_{ij}^*) \epsilon_{ij}^{irr}} - \frac{1}{2} \overline{(\sigma_{ij}^{Rirr} + \sigma_{ij}^*) a_{ij}^2 \Delta T} \\
&+ \frac{1}{2} \overline{a_{ijkl}^1 b_{ij} b_{kl} (\Delta T^*)^2} - \frac{1}{2} \overline{b_{ij} \Delta T^* a_{ij}^{2*} (T^* - T^\circ)} \\
&- \frac{1}{2} \overline{a_{ijkl}^1 [(a_{ijmn}^1)^{-1} - (a_{ijmn}^{1*})^{-1}] (\sigma_{kl}^{Rirr} + b_{kl} \Delta T^* + \sigma_{kl}^*) [\epsilon_{mn}^{irr*} + a_{mn}^{2*} (T^* - T^\circ)]}.
\end{aligned} \tag{359}$$

In this case the macroscopic entropy, \bar{s} , is

$$\begin{aligned}
\bar{\rho} \bar{s} &= -\bar{\rho} \frac{\partial \overline{\rho G}}{\partial T} = -\frac{\partial (\overline{\rho G})}{\partial T} = A_{ij}^2 \Sigma_{ij} + \rho a^3 \ln \left(\frac{T}{T^\circ} \right) + \overline{\rho a^4} - \frac{1}{2} \overline{\epsilon_{ij}^{irr} b_{ij}} + \frac{1}{2} \overline{(\sigma_{ij}^{Rirr} + \sigma_{ij}^*) a_{ij}^{2*}} \\
&- \overline{a_{ijkl}^1 b_{ij} b_{kl} \Delta T^*} + \frac{1}{2} \overline{a_{ijkl}^1 [(a_{ijmn}^1)^{-1} - (a_{ijmn}^{1*})^{-1}] b_{kl} [\epsilon_{mn}^{irr*} + a_{mn}^{2*} (T^* - T^\circ)]}.
\end{aligned} \tag{360}$$

In this case the macroscopic entropy, \bar{s} , is

$$\begin{aligned}
\bar{\rho} \bar{s} &= -\bar{\rho} \frac{\partial \overline{\rho G}}{\partial T} = -\frac{\partial (\overline{\rho G})}{\partial T} = A_{ij}^2 \Sigma_{ij} + \rho a^3 \ln \left(\frac{T}{T^\circ} \right) + \overline{\rho a^4} - \frac{1}{2} \overline{\epsilon_{ij}^{irr} b_{ij}} + \frac{1}{2} \overline{(\sigma_{ij}^{Rirr} + \sigma_{ij}^*) a_{ij}^{2*}} \\
&- \overline{a_{ijkl}^1 b_{ij} b_{kl} \Delta T^*} + \frac{1}{2} \overline{a_{ijkl}^1 [(a_{ijmn}^1)^{-1} - (a_{ijmn}^{1*})^{-1}] b_{kl} [\epsilon_{mn}^{irr*} + a_{mn}^{2*} (T^* - T^\circ)]}.
\end{aligned} \tag{360}$$

$$\begin{aligned}
\bar{\rho} \hat{G}^{th \text{ pure}} \{T, c^{Mv}\} &= -\rho a^3 \left[\Delta T - T \ln \left(\frac{T}{T^\circ} \right) \right] - \overline{\rho a^4} T + \overline{\rho a^5} \\
\bar{\rho} \hat{G}^{th \text{ mix}} \{T, c^{Mv}\} &= \frac{1}{2} \overline{a_{ijkl}^1 b_{ij} b_{kl} (\Delta T^*)^2} - \frac{1}{2} \overline{b_{ij} a_{ij}^{2*} (T^* - T^\circ) \Delta T^*} - \frac{1}{2} \overline{\sigma_{ij}^* a_{ij}^{2*} \Delta T} \\
&- \frac{1}{2} \overline{a_{ijkl}^1 [(a_{ijmn}^1)^{-1} - (a_{ijmn}^{1*})^{-1}] (b_{kl} \Delta T^* + \sigma_{kl}^*) a_{mn}^{2*} (T^* - T^\circ)}
\end{aligned} \tag{405a - b}$$

and

$$\begin{aligned}
\bar{\rho} \hat{G}^{eig \text{ pure}} \{c^{Mv}, \epsilon_{ij}^{irr v}\} &= \frac{1}{2} \overline{\rho b^7 (c^M)^2} + \frac{1}{2} \overline{\rho b^4 \epsilon_{ij}^{irr} \epsilon_{ij}^{irr}} \\
\bar{\rho} \hat{G}^{eig \text{ mix}} \{T, c^{Mv}, \epsilon_{ij}^{irr v}\} &= \frac{1}{2} \overline{(\sigma_{ij}^{Rirr} + b_{ij} \Delta T^* + \sigma_{ij}^*) \epsilon_{ij}^{irr}} - \frac{1}{2} \overline{\sigma_{ij}^{Rirr} a_{ij}^{2*} \Delta T} - \frac{1}{2} \overline{b_{ij} a_{ij}^{2*} (T^* - T^\circ)} \\
&- \frac{1}{2} \overline{a_{ijkl}^1 [(a_{ijmn}^1)^{-1} - (a_{ijmn}^{1*})^{-1}] (\sigma_{kl}^{Rirr} + b_{kl} \Delta T^* + \sigma_{kl}^*) \epsilon_{mn}^{irr*}} \\
&- \frac{1}{2} \overline{a_{ijkl}^1 [(a_{ijmn}^1)^{-1} - (a_{ijmn}^{1*})^{-1}] \sigma_{kl}^{Rirr} a_{mn}^{2*} (T^* - T^\circ)}
\end{aligned} \tag{406a - b}$$

$$\frac{\partial B_{ijkl}^f}{\partial c^{Mm}} = -B_{ijrs}^f c^m (I_{mnop} - S_{mnop}) \left[(a_{abrs}^{1m})^{-1} \Delta a_{abfg}^{1m} (a_{fgmn}^{1m})^{-1} (a_{opuv}^{1f} - a_{opuv}^{1m}) \right. \\ \left. + (a_{rsmn}^{1m})^{-1} \Delta a_{opuv}^{1m} \right] B_{uvkl}^f. \quad (852)$$

$$\frac{\partial B_{ijkl}^m}{\partial c^{Mf}} = ??? \quad \frac{\partial B_{ijkl}^m}{\partial c^{Mm}} = ??? \quad (853)$$

$$\frac{\partial \mathcal{P}_{ijmn}^f}{\partial c^{Mf}} = \left[-\frac{\partial B_{ijfg}^f}{\partial c^{Mf}} + (I_{ijkl} - B_{ijkl}^f) (a_{stkl}^{1m} - a_{stkl}^{1f})^{-1} \Delta a_{stfg}^{1f} \right] (a_{fgmn}^{1m} - a_{fgmn}^{1f})^{-1} \quad (854)$$

$$\frac{\partial \mathcal{P}_{ijmn}^f}{\partial c^{Mm}} = - \left[\frac{\partial B_{ijfg}^f}{\partial c^{Mm}} + (I_{ijkl} - B_{ijkl}^f) (a_{stkl}^{1m} - a_{stkl}^{1f})^{-1} \Delta a_{stfg}^{1m} \right] (a_{fgmn}^{1m} - a_{fgmn}^{1f})^{-1} \quad (855)$$

$$\frac{\partial \mathcal{P}_{ijmn}^m}{\partial c^{Mf}} = \left[-\frac{\partial B_{ijfg}^m}{\partial c^{Mf}} + (I_{ijkl} - B_{ijkl}^m) (a_{stkl}^{1m} - a_{stkl}^{1f})^{-1} \Delta a_{stfg}^{1f} \right] (a_{fgmn}^{1m} - a_{fgmn}^{1f})^{-1} \quad (856)$$

$$\frac{\partial \mathcal{P}_{ijmn}^m}{\partial c^{Mm}} = - \left[\frac{\partial B_{ijfg}^m}{\partial c^{Mm}} + (I_{ijkl} - B_{ijkl}^m) (a_{stkl}^{1m} - a_{stkl}^{1f})^{-1} \Delta a_{stfg}^{1m} \right] (a_{fgmn}^{1m} - a_{fgmn}^{1f})^{-1} \quad (857)$$

The rate of change of the irreversible eigenstress is

$$\dot{\sigma}_{ij}^{Rirr} = \left(\frac{\partial \mathcal{P}_{ijmn}^v}{\partial c^{Mf}} \dot{c}^{Mf} + \frac{\partial \mathcal{P}_{ijmn}^v}{\partial c^{Mm}} \dot{c}^{Mm} \right) (\epsilon_{mn}^{Dirrf} - \epsilon_{mn}^{Dirrm}) \\ + \mathcal{P}_{ijmn}^v (\Lambda_{mn}^f \dot{c}^{Mf} + \dot{\epsilon}_{mn}^{rf} - \Lambda_{mn}^m \dot{c}^{Mm} - \dot{\epsilon}_{mn}^{rm}). \quad (858)$$

THIS NEEDS TO BE REDONE !!! Therefore, the rate of change of the fiber thermal eigenstress is

$$\dot{\sigma}_{ij}^{Rthf} = \frac{\partial b_{ij}^f}{\partial c^M} \dot{c}^M + b_{ij}^f \dot{T} \quad (1)$$

where

$$\frac{\partial b_{ij}^f}{\partial c^M} = - \frac{\partial B_{ijkl}^f}{\partial c^M} (a_{klmn}^{1m} - a_{klmn}^{1f})^{-1} (a_{mn}^{2f} - a_{mn}^{2m}) \\ + \mathcal{P}_{ijpq}^f \Delta a_{pqrs}^{1f} (a_{rsmn}^{1m} - a_{rsmn}^{1f})^{-1} (a_{mn}^{2f} - a_{mn}^{2m}) + \mathcal{P}_{ijpq}^f \Delta a_{mn}^{2f} \quad (1)$$

Micromechanical methods of predicting the effective properties of elastic composites include simple bounding methods (VOIGT, 1889; REUSS, 1929), improved bounding methods (HASHIN and SHTRICKMAN [1962, 1963]; HILL [1963]; WALPOLE [1966a, 1966b], the classical self-consistent scheme (KRONER [1958]; BUDIANSKY and WU [1962]), the generalized self-consistent scheme (CHRISTENSEN and LO [1979], and the MORI-TANAKA [1973] method. The improved bounding methods use variational principles that require the stress-strain response to be derivable from true potential functions. These methods are therefore of limited use in modelling inelastic composites because inelasticity usually results in path-dependent material behavior. The self-consistent

method is ideally suited for the elastic and inelastic (HILL [1965] behavior of materials, such as polycrystals, in which there is not a clear distinction between the matrix and the reinforcing phase. Although the generalized self-consistent scheme has been applied to fibrous metal matrix composites (DVORAK and RAO [1976], the Mori-Tanaka method is computationally more efficient because it is an explicit method, whereas the self-consistent methods are implicit. In fact, the Mori-Tanaka method is now widely used in the study of inelastic composites (ARSENAULT and TAYA [1987]; TANDON and WENG [1988]; LAGOUDAS, GAVAZZI, and NIGAM [1991]. A common element in the self-consistent schemes and the Mori-Tanaka method is the use of ESHELBY's [1957] solution of a single inclusion in an infinitely extended effective medium or matrix material, respectively.